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INTRODUCTION TO THE GROUP THEORY OF ELEMENTARY PARTICLES

BY

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PREFACE

This communication had its origin in seminars on Lie algebras, with special reference to the B_2 group, given in the Spring of 1964 at the Physics Department, Laval University, Quebec. In seminars for theoretical and experimental physicists given at the Dublin Institute for Advanced Studies during the Winter of 1964-65 the subject-matter was re-ordered and expanded so as to include the SU_3 group and its chief experimental implications.

James McConnell

Maynooth,

April, 1965.

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CHAPTER I:

The Elementary Particles.

It may be claimed that the modern theory of elementary particles was initiated in Dublin in February 1881 when Johnstone Stoney presented to the Royal Dublin Society a paper in which he put forward the idea that there exists a unit of electrical charge, both positive and negative[1]. The unit of negative electrical charge was later named by Stoney the <u>electron</u> and in 1897 it was detected experimentally by J. J. Thomson. It is customary nowadays to express mass in terms of energy employing Einstein's relation that a particle of mass m has rest energy $m c^2$, and the mass of the electron is 0.51 MeV. The rays which come from the cathode of a vacuum tube consist of electrons. The canal rays that go to the cathode consist of positively charged particles; those that come from hydrogen are called <u>protons</u>. The mass of the proton is 938 MeV — about 1836 times that of the electron.

In 1900 Planck introduced the quantum theory that energy is increased or diminished by units $h\nu$, where ν is the frequency of the radiation, and five years later Einstein proposed that electromagnetic radiation consists of particles called <u>photons</u> which have energy $h\nu$ and velocity v equal to c. The rest mass of these particles must be zero in order that we may have finite energy $m c^2 (1 - v^2/c^2)^{-\frac{1}{2}}$.

We now go on about twenty-five years. In the meantime quantum mechanics had been created by Heisenberg and Schrödinger, and Dirac had

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(1) G. Johnstone Stoney, <u>Sc. Proc. Roy. Dublin Soc. 3</u>, 51 (1881).

succeeded in obtaining a relativistic equation for the electron. The relativistic formula

$$E^2 = m^2 c^4 + c^2 p^2$$

gives positive and negative values of the energy, and Dirac interpreted the negative energy states of the electron as referring to particles with positive charge. At first he thought that this must be the proton but in 1931 he confessed that it must be a particle with the mass of the electron but with positive electrical charge. This particle — the positron — was discovered in the following year by C. D. Anderson. This momentous discovery started a new line in thinking. It gave us the idea of antimatter, that to every charged particle there should exist an antiparticle with the same mass but opposite charge. Also in 1932 the neutron was discovered by Chadwick. This is an uncharged particle with mass slightly greater than that of the proton. Protons and neutrons form the constituents of the atomic nucleus.

The first example of a particle with mass intermediate between that of the electron and the proton was discovered in 1934 by Kunze. This particle is now called the muon. Its properties are very like those of the electron, it can be positively or negatively charged, but its mass is 105.7 MeV — more than two hundred times that of the electron — and it decays after 2.2×10^{-6} sec.

More than ten years were to pass before the next elementary particle was discovered. In 1946 Occhialini and Powell discovered the pion which can be positively or negatively charged, or uncharged. The mass of the charged pion is 139.6 MeV and its lifetime is 2.5×10^{-8} sec; the mass of the neutral pion is somewhat less, viz. 135 MeV, and its lifetime is

1.8×10^{-16} sec. The pion is **very** important. It is the quantum of the forces between protons and neutrons in the nucleus.

The particle now known as the <u>kaon</u> was first discovered by Rochester and Butler in 1948. Several years of patient work were required to systematize the properties of kaons. Ignoring some very recent results on the decay of kaons[2)] we may present the picture as follows: there are the K^+ and its antiparticle K^- with mass 493.8 MeV and lifetime 1.29×10^{-8} sec, and the K^0 and its distinct antiparticle $\bar{K}^0$ with mass 498 MeV. These latter have no precise lifetimes but there are two mixed states of them — K_1^0 with lifetime 0.92×10^{-10} and K_2^0 with lifetime 5.62×10^{-8} sec.

With the advent of the 1950's things happened in quick succession and we shall not attempt to preserve chronological order. Already in the 1930's Pauli had suggested that in the decay of a neutron into a proton and electron a neutral particle with spin $\frac{1}{2}$ takes part. This particle was called the <u>neutrino</u> and it was discovered by Reines and Cowan in 1955. It seems to have zero rest mass. Anticipating further results we say that there are two neutrinos: one ν_e associated with the electron and the other ν_μ associated with the muon, and that moreover the neutrino though uncharged differs from the antineutrino.

About this time there were discovered particles called <u>hyperons</u> with masses greater than that of the neutron. They are the Λ^0 with mass 1115 MeV and lifetime 2.6×10^{-10} sec; the Σ-particles, Σ^+ with mass 1189 MeV and lifetime 0.79×10^{-10} sec, the Σ^0 with mass 1192 MeV and lifetime less than 10^{-14} sec. and the Σ^- with mass 1197 MeV and lifetime 1.6×10^{-10} sec; the cascade particles, Ξ^0 with mass 1314 MeV

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(2) J. H. Christenson, J. W. Cronin, V. L. Fitch and R. Turley, <u>Phys. Rev. Lett.</u> <u>13</u>, 138 (1964).

and lifetime 3.06×10^{-10} sec. and Ξ^- with mass 1321 MeV and lifetime 1.74×10^{-10} sec.

There are broader classifications of elementary particles: the neutrinos, electrons, and muons are called <u>leptons</u>. The pions and kaons are called <u>mesons</u>. The protons and neutrons are called <u>nucleons</u>; the nucleons and hyperons are called <u>baryons</u>. The kaons and hyperons are called <u>strange particles</u>. Antiparticles of all these particles have been found.

In recent years the high-energy accelerators have provided an abundance of a new type of particle called a <u>resonance</u>. These decay rapidly into the elementary particles listed above; for example, the η with mass 549 MeV may decay into 3 pions, the $N^*_{\frac{1}{2}}$ with mass 1480 MeV decays into N and π.

It would be nice at this stage to give a table of elementary particles. To do this adequately we must distinguish the different types of interaction that can occur between elementary particles: (a) strong interactions, (b) electromagnetic interactions, (c) weak interactions, (d) gravitational interactions.

<u>Strong interactions</u> are those responsible for nuclear forces, the production of pions, of strange particles and of resonances. The <u>strongly interacting particles</u>, also called <u>hadrons</u>, are nucleons, mesons, hyperons, and their antiparticles.

<u>Electromagnetic interactions</u> are responsible for processes like photoelectric effect, Compton effect, and for mass differences within <u>isotopic multiplets</u> of strongly interacting particles. Examples of isotopic multiplets are $N\begin{pmatrix} p \\ n \end{pmatrix}$, $\pi\begin{pmatrix} \pi^+ \\ \pi^o \\ \pi^- \end{pmatrix}$, $K\begin{pmatrix} K^+ \\ K^o \end{pmatrix}$, $\Xi\begin{pmatrix} \Xi^o \\ \Xi^- \end{pmatrix}$. If we have two members, we say that the isospin is $\frac{1}{2}$ and that the third component I_3 takes the values $\frac{1}{2}$, $-\frac{1}{2}$, e.g. $I_3 = \frac{1}{2}$ for p and $I_3 = -\frac{1}{2}$ for n: if we

have three members, we say that the isospin is 1, and that I_3 assumes the values 1, 0, -1, e.g. $I_3 = 1$ for π^+, 0 for π^0 and -1 for π^-. For an antiparticle the I_3-eigenvalue is minus that of the particle, e.g. $I_3 = -\frac{1}{2}$ for $\bar{p}$.

Weak interactions are those responsible for decay processes.

Gravitational interactions are very much weaker than any of the above, and we shall ignore them.

Let us say a little about quantities that are conserved in these interactions. Of course, we have always conservation of energy, linear momentum, angular momentum and charge. Another quantity that is always conserved is baryon number.

Baryon number B is defined as the sum of $+1$ for each baryon and as -1 for each antibaryon. If Q is the charge of a strongly interacting particle, its hypercharge Y is defined by the relation $Q = I_3 + \frac{1}{2}Y$. For a proton (neutron) Q is 1 (0) and I_3 is $\frac{1}{2}$ ($-\frac{1}{2}$), so for both of them $Y = 1$. For π, $Q = I_3$ so $Y = 0$, for Ξ^0 (Ξ^-) Q is 0 (-1) and I_3 is $\frac{1}{2}$ ($-\frac{1}{2}$) so Y is -1 for Ξ. Strangeness S is defined by $Y = B + S$. It is non-vanishing only for strange particles. We see that $S = 1$ for K $\binom{K^+}{K^0}$, $S = -1$ for $\bar{K}$ $\binom{K^-}{\bar{K}^0}$, $S = -1$ for Λ and Σ, $S = -2$ for Ξ. Hypercharge and strangeness change sign for an antiparticle. I_3 and Y, or I_3 and S, are conserved in strong and electromagnetic interactions. In addition total isospin I is conserved in strong interactions.

There are conservation laws also for weak interactions. We have conservation of lepton number and of muon number. The lepton number is $+1$ for the leptons e^-, μ^-, ν_e, ν_μ and -1 for the antileptons e^+, μ^+,

$\bar{\nu}_e$, $\bar{\nu}_\mu$ and in any process the lepton number of the particles present must be the same before and after. The muon number is $+1$ for μ^- and ν_μ , and -1 for μ^+ and $\bar{\nu}_\mu$, zero for the others, and we must have the same muon number before and after an interaction.

We can now list some of the properties of elementary particles[3].

Class	Particle	Mass (MeV)	Spin	Muon Number	I_3	Y	S	Lifetime (Sec.)	Anti-particle
photon	γ	0	1					stable	γ
leptons	ν_e	< .0002	$\frac{1}{2}$	0				stable	$\bar{\nu}_e$
	ν_μ	< 4	$\frac{1}{2}$	1				stable	$\bar{\nu}_\mu$
	e^-	.51	$\frac{1}{2}$	0				stable	e^+
	μ^-	105.7	$\frac{1}{2}$	1				2.2×10^{-6}	μ^+
mesons	π: π^+	139.6	0		1	0	0	2.6×10^{-8}	π^-
	π: π^0	135	0		0	0	0	1.8×10^{-16}	π^0
	π: π^-	139.6	0		-1	0	0	2.6×10^{-8}	π^+
	K: K^+	493.8	0		$\frac{1}{2}$	1	1	1.23×10^{-8}	K^-
	K: K^0 $\begin{cases} K^0_1 \\ K^0_2 \end{cases}$	498	0		$-\frac{1}{2}$	1	1	0.92×10^{-10} 5.62×10^{-8}	$\bar{K}_0$ $\begin{cases} K^0_1 \\ K^0_2 \end{cases}$
baryons	N: p	938	$\frac{1}{2}$		$\frac{1}{2}$	1	0	stable	$\bar{p}$
	N: n	939.5	$\frac{1}{2}$		$-\frac{1}{2}$	1	0	$1.01 \times 10^{+3}$	$\bar{n}$
hyperons	Λ^0	1115	$\frac{1}{2}$		0	0	-1	2.62×10^{-10}	$\overline{\Lambda^0}$
	Σ: Σ^+	1189	$\frac{1}{2}$		1	0	-1	0.79×10^{-10}	$\overline{\Sigma^+}$
	Σ: Σ^0	1192	$\frac{1}{2}$		0	0	-1	$< 10^{-14}$	$\overline{\Sigma^0}$
	Σ: Σ^-	1197	$\frac{1}{2}$		-1	0	-1	1.58×10^{-10}	$\overline{\Sigma^-}$
	Ξ: Ξ^0	1314	$\frac{1}{2}$		$\frac{1}{2}$	-1	-2	3.1×10^{-10}	$\overline{\Xi^0}$
	Ξ: Ξ^-	1321	$\frac{1}{2}$		$-\frac{1}{2}$	-1	-2	1.74×10^{-10}	$\overline{\Xi^-}$

(3) cf. A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz and M. Roos, Rev. Mod. Phys. 36, 977 (1964).

We notice that the only particle with mechanical spin one is the photon, that all the leptons have spin $\frac{1}{2}$, that of the strongly interacting particles the mesons have spin 0 and the baryons spin $\frac{1}{2}$. All the above particles are sometimes called "stable particles", the criterion being that their life-times are long compared with 10^{-23} sec.

Let us arrange the baryons according to their (I_3 , Y) values. We have eight positions as shown, those of Λ^0 and Σ^0 being coincident. The diagram includes the multiplets (p, n) , $(\Sigma^+, \Sigma^0, \Sigma^-)$, (Ξ^0, Ξ^-) and is called a <u>supermultiplet</u>. It is in this case an <u>octet</u>. It might be remarked that there seems to be a similar octet picture for resonances with spin 5/2.

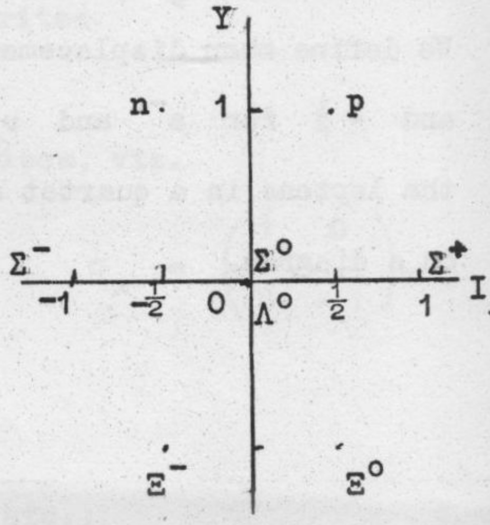

Fig 1 The nucleon octet.

Next let us arrange the mesons. There are seven of them but, if we want to have an isotopic singlet like the Λ^0 , we can add the resonance η^0 and obtain an octet. We notice that both particles and antiparticles are on the diagrams. This is not surprising because we have no law of conservation of meson number, like that of baryon number. ~~Again~~ There is ~~some evidence of~~ a similar octet picture of resonances of spin 1 .

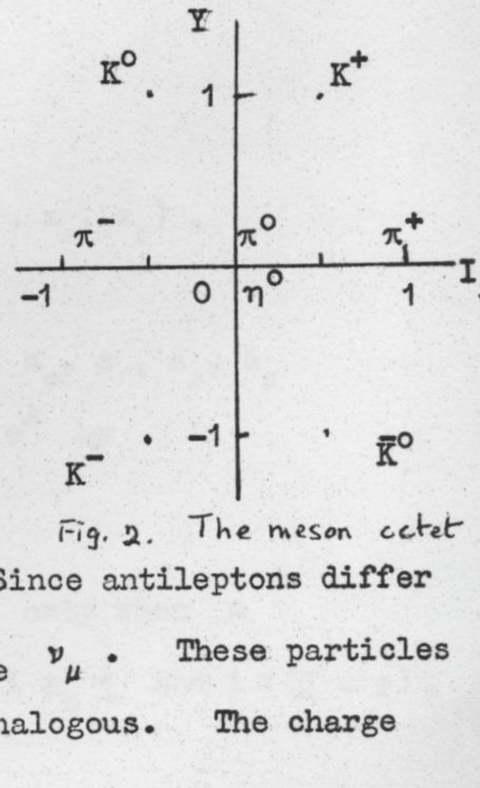

Fig. 2. The meson octet

Finally let us construct a diagram for leptons. Since antileptons differ from leptons, we just want to accommodate e^- , μ^- , ν_e ν_μ . These particles have no (I_3 , Y) values but we can define something analogous. The charge of a lepton is 0 or -1. We define charge displacement as a two-component quantity having the values $+\frac{1}{2}$ for charge zero and $-\frac{1}{2}$ for charge -1. The muon number is likewise $+1$ for μ^- and ν_μ and zero for e^- and ν_e. We define muon displacement as $\frac{1}{2}$ for μ^- and ν_μ, and $-\frac{1}{2}$ for e^- and ν_e. We may then arrange the leptons in a quartet at the points $(\pm\frac{1}{2}, \pm\frac{1}{2})$ on a diagram.

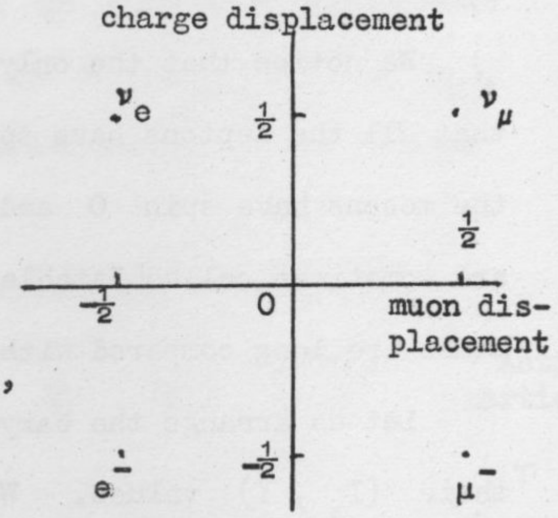

Fig. 3. The Lepton quartet.

CHAPTER II:

The SU_2 Group.

In this chapter we recall some points from the Pauli theory of spin, and we express the theory in the language of groups. To describe the spin angular momentum $\underset{\sim}{m}$ of an electron one writes

$$\underset{\sim}{m} = \tfrac{1}{2}\hbar\underset{\sim}{\sigma} ,$$

where $\underset{\sim}{\sigma}$ denotes the triad $(\sigma_1, \sigma_2, \sigma_3)$ of matrices, viz.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that

$$\sigma_1^{*} = \sigma_1 , \quad \sigma_1^2 = 1 , \text{ etc.,}$$

$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = 0 , \quad \sigma_2\sigma_3 = i\sigma_1 , \text{ etc.,}$$

where 1 is the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Any two-by-two matrix a has four independent elements, so it may be expressed as a linear combination of $1, \sigma_1, \sigma_2, \sigma_3$. We write

$$\begin{aligned} a &= a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \\ &= a_0 1 + \alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e}) , \end{aligned}$$

where $\underset{\sim}{e}$ is a unit vector in the direction of (a_1, a_2, a_3),

$$\alpha^2 = a_1^2 + a_2^2 + a_3^2$$

and we shall be concerned only with the case where a_0, a_1, a_2, a_3 are all real. If A is any operator, we define e^A by

$$e^A = 1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

and we notice that e^{A+B} is equivalent to $e^A e^B$ only when A and B commute. This condition is satisfied by $i\, a_0 1$ and $i\,\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})$, so

$$e^{i\alpha} = e^{i a_0 \mathbb{1}} e^{i\alpha(\sigma . e)} .$$

Now

$$e^{i a_0 \mathbb{1}} = \mathbb{1} + i a_0 \mathbb{1} + \frac{(i a_0)^2}{2!}\mathbb{1} + \frac{(i a_0)^3}{3!}\mathbb{1} + \dots = e^{i a_0} \mathbb{1}$$

$$\begin{aligned}(\underset{\sim}{\sigma} . \underset{\sim}{e})^2 &= (\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3)(\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3) \\ &= \sigma_1{}^2 e_1{}^2 + \sigma_2{}^2 e_2{}^2 + \sigma_3{}^2 e_3{}^2 + (\sigma_2\sigma_3 + \sigma_3\sigma_2) e_2 e_2 + \dots + \dots \\ &= \mathbb{1}\end{aligned}$$

$$(\underset{\sim}{\sigma} . \underset{\sim}{e})^3 = (\underset{\sim}{\sigma} . \underset{\sim}{e}) \quad \text{etc.}$$

and we may write

$$e^{i\alpha} = e^{i a_0} e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})}$$

$$e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} = \mathbb{1} \cos\alpha + i\,(\underset{\sim}{\sigma} . \underset{\sim}{e}) \sin\alpha . \qquad (2.1)$$

Written out in full

$$e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} = \begin{pmatrix} \cos\alpha + i\, e_3 \sin\alpha & i\, e_1 \sin\alpha + e_2 \sin\alpha \\ i\, e_1 \sin\alpha - e_2 \sin\alpha & \cos\alpha - i\, e_3 \sin\alpha \end{pmatrix}$$

We see immediately that the determinant of $e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})}$ is unity; that is to say, $e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})}$ is a <u>unimodular matrix</u>. It is also unitary because according to (2.1) its hermitian conjugate is $e^{-i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})}$ which is just its inverse, since

$$e^{-i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} = (\mathbb{1} \cos\alpha - i\,(\underset{\sim}{\sigma} . \underset{\sim}{e}) \sin\alpha)(\mathbb{1} \cos\alpha + i\,(\underset{\sim}{\sigma} . \underset{\sim}{e}) \sin\alpha) = \mathbb{1}$$

$$e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} e^{-i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} = \mathbb{1} .$$

If f denotes a two-component spinor $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, then $e^{i\alpha(\underset{\sim}{\sigma} . \underset{\sim}{e})} f$ may be interpreted[4] as the spinor rotated about $\underset{\sim}{e}$ through an angle 2α .

(4) cf. J. McConnell, <u>Annali di Matematica</u> <u>57</u>, 203 (1962).

It is clear that e^{ia} is unitary, but it is unimodular only when a_o vanishes.

A <u>group</u> may be defined as a set of operations with the following properties:-

(a) there exists the identity,

(b) for every operation there exists an inverse operation,

(c) the product of two operations is a member of the set.

The set of unimodular unitary operations given by the matrices $e^{i\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})}$ constitutes a group: there exists the identity obtained by putting α equal to zero, to each $e^{i\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})}$ there exists the inverse $e^{-i\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})}$. Lastly the product of two unitary matrices is a unitary matrix and the determinant of the product is the product of the determinants, so the product of two unimodular unitary transformations is a unimodular unitary transformation. The group is the <u>unimodular unitary group</u> in two dimensions and it is denoted by SU_2. Like all unitary groups in two dimensions, denoted by U_2, it leaves invariant $f_1{}^* f_1 + f_2{}^* f_2$. In fact this sum is just $f^+ f$ and, if we put

$$f' = e^{i\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})} f$$

so that

$$f'^+ = f^+ e^{-i\alpha(\underset{\sim}{\sigma}\cdot\underset{\sim}{e})}$$

we obtain

$$f'^+ f' = f^+ f .$$

A group is said to be <u>Abelian</u>, if the order in which we perform two operations is immaterial. If we perform two SU_2 operations with

$$1 \cos\alpha + i\,(\underset{\sim}{\sigma}\cdot\underset{\sim}{e}) \sin\alpha , \qquad 1 \cos\alpha' + i\,(\underset{\sim}{\sigma}\cdot\underset{\sim}{e}') \sin\alpha' ,$$

the two operations do not commute because $(\underset{\sim}{\sigma}.\underset{\sim}{e})$ and $(\underset{\sim}{\sigma}.\underset{\sim}{e}')$ do not commute. Hence SU_2 is non-Abelian.

An <u>infinitesimal transformation</u> is obtained by neglecting terms proportional to α^2, α^3 etc. in the expansion of $e^{i\alpha(\underset{\sim}{\sigma}.\underset{\sim}{e})}$.

Let us write

$$\begin{aligned} U &= 1 + i\,\alpha\,(\underset{\sim}{\sigma}.\underset{\sim}{e}) \\ &= 1 + i\,(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) \\ &= 1 + i\,\varepsilon^A\sigma_A\,, \end{aligned}$$

where ε^A are infinitesimals and there is a summation from 1 to 3 over the repeated index A. If a labels a row and b a column,

$$U_a{}^b = \delta_a{}^b + i\,\varepsilon^A\,\sigma_{Aa}{}^b\,,$$

where $\delta_a{}^b$ is the Kronecker delta equal to unity for $b = a$ and equal to zero otherwise. To a first approximation the infinitesimal transformations form an Abelian group because

$$\begin{aligned} (1 + i\varepsilon^A\sigma_A)\,(1 + i\varepsilon'^B\sigma_B) &= 1 + i\,(\varepsilon^A + \varepsilon'^A)\sigma_A \\ &= (1 + i\varepsilon'^B\sigma_B)\,(1 + i\varepsilon^A\sigma_A)\,. \end{aligned}$$

By iterating the infinitesimal transformation we can obtain a finite transformation. To see this we take a finite angle θ and a large positive integer s and we write θ/s for α, so that

$$U = 1 + \frac{i\,\theta(\underset{\sim}{\sigma}.\underset{\sim}{e})}{s}\,.$$

Then

$$U^s = \left(1 + i\,\frac{\theta(\underset{\sim}{\sigma}.\underset{\sim}{e})}{s}\right)^s$$

and

$$\lim_{s\to\infty} U^s = e^{i\theta(\underset{\sim}{\sigma}.\underset{\sim}{e})}$$

which is the finite transformation.

We now define two linear combinations σ_+, σ_- of the Pauli matrices σ_1, σ_2 by

$$\sigma_+ = \tfrac{1}{2}(\sigma_1 + i\,\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = \tfrac{1}{2}(\sigma_1 - i\,\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_+{}^+ .$$

Since

$$\sigma_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sigma_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \sigma_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

we interpret σ_+ as a <u>raising operator</u>, that raises a state with $m_3 = -\frac{1}{2}\hbar$ to a state with $m_3 = \frac{1}{2}\hbar$, and we interpret σ_- as a <u>lowering operator</u>. From the relations obeyed by σ_1, σ_2, σ_3 we deduce

$$[\sigma_+ , \sigma_-] = \frac{1}{4}\{(\sigma_1 + i\sigma_2)(\sigma_1 - i\sigma_2) - (\sigma_1 - i\sigma_2)(\sigma_1 + i\sigma_2)\} = \sigma_3$$

$$[\sigma_3, \sigma_+] = 2\,\sigma_+ , \qquad [\sigma_3, \sigma_-] = -2\,\sigma_- .$$

Henceforth we shall often write the two-dimensional wave functions $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $|\{2\}, 1\rangle$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $|\{2\}, 2\rangle$. Then

$$\langle 1, \{2\}| = |\{2\}, 1\rangle^{*} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\langle 2, \{2\}| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\langle 1, \{2\}|\{2\}, 1\rangle = 1 , \qquad \langle 1, \{2\}|\{2\}, 2\rangle = 0 \tag{2.3}$$

$$\langle 2, \{2\}|\{2\}, 1\rangle = 0 , \qquad \langle 2, \{2\}|\{2\}, 2\rangle = 1 ,$$

and equations (2.2) read

$$\begin{aligned} \sigma_{+}|\{2\}, 2\rangle &= |\{2\}, 1\rangle , \qquad & \sigma_{+}|\{2\}, 1\rangle &= 0 \\ \sigma_{-}|\{2\}, 2\rangle &= 0 , & \sigma_{-}|\{2\}, 1\rangle &= |\{2\}, 2\rangle . \end{aligned} \tag{2.4}$$

Where there is no danger of confusion we may find it convenient to drop the $\{2\}$ and write the wave functions as $|1\rangle$ and $|2\rangle$.

Let us examine the spin functions of a two-electron system. These are products, or sums of products, of $|\{2\}, 1\rangle$ and $|\{2\}, 2\rangle$. When we write down $|\{2\}, a\rangle |\{2 , b\rangle$, we shall understand that $|\{2\}, a\rangle$ refers to the first electron and $|\{2\}, b\rangle$ to the second, and we note that $|a\rangle|b\rangle$ differs from $|b\rangle|a\rangle$ when $b \neq a$. There will be spin matrices $\boldsymbol{\sigma}(1)\ (\sigma_1(1), \sigma_2(1), \sigma_3(1))$ referring to the first electron and $\boldsymbol{\sigma}(2)\ (\sigma_1(2), \sigma_2(2), \sigma_3(2))$ referring to the second. In the case of the product $|a\rangle|b\rangle$, $\boldsymbol{\sigma}(1)$ acts on $|a\rangle$ but leaves $|b\rangle$ unchanged; that is to say, it is just the unit matrix in the spin space of the second electron. By $\boldsymbol{\sigma}$ we shall understand the sum of the two operators $\boldsymbol{\sigma}(1)$ and $\boldsymbol{\sigma}(2)$, and

$$\mathbf{m} = \tfrac{1}{2}\hbar\boldsymbol{\sigma} = \tfrac{1}{2}\hbar(\boldsymbol{\sigma}(1) + \boldsymbol{\sigma}(2)).$$

Thus

$$\begin{aligned} \sigma_3|1\rangle|1\rangle &= (\sigma_3(1) + \sigma_3(2))\,|1\rangle|1\rangle = \sigma_3(1)\,|1\rangle \,.\, |1\rangle + \\ &\qquad + |1\rangle \,.\, \sigma_3(2)\,|1\rangle \\ &= |1\rangle\,|1\rangle + |1\rangle\,|1\rangle = 2\,|1\rangle\,|1\rangle , \end{aligned}$$

so that $|1\rangle|1\rangle$ is an eigenstate of m_3 with eigenvalue $\hbar$. In the same way equations (2.4) yield

$$\sigma_-|1\rangle|1\rangle = |2\rangle|1\rangle + |1\rangle|2\rangle, \quad m_3(|2\rangle|1\rangle + |1\rangle|2\rangle) = 0$$
$$\sigma_-(|2\rangle|1\rangle + |1\rangle|2\rangle) = 2|2\rangle|2\rangle, \quad m_3|2\rangle|2\rangle = -\hbar|2\rangle|2\rangle$$
$$\sigma_-|2\rangle|2\rangle = 0, \quad \sigma_+|1\rangle|1\rangle = 0.$$

So by applying raising and lowering operators we obtain a set of three, and only three, normalized spin functions

$$|\{3\},1\rangle = |\{2\},1\rangle|\{2\},1\rangle$$
$$|\{3\},2\rangle = \frac{1}{\sqrt{2}}(|\{2\},1\rangle|\{2\},2\rangle + |\{2\},2\rangle|\{2\},1\rangle) \quad (2.5)$$
$$|\{3\},3\rangle = |\{2\},2\rangle|\{2\},2\rangle$$

with m_3 eigenvalues $\hbar$, 0, $-\hbar$, respectively. They are orthogonal to one another, as we see by employing (2.3). However there is another spin function $\frac{1}{\sqrt{2}}(|1\rangle|2\rangle - |2\rangle|1\rangle)$ orthogonal to the functions in (2.5), having m_3 eigenvalue zero and being annihilated by σ_+ and σ_-. It therefore represents a <u>singlet</u> spin zero state, for which we write

$$|\{1\},1\rangle = \frac{1}{\sqrt{2}}(|\{2\},1\rangle|\{2\},2\rangle - |\{2\},2\rangle|\{2\},1\rangle,$$

while (2.5) are the members of a <u>triplet</u> spin 1 state.

The above procedure may be pictured geometrically by drawing vectors representing the eigenvalues $\pm\frac{1}{2}$ of $\frac{1}{2}\sigma_3(1)$ and $\frac{1}{2}\sigma_3(2)$. Compounding the vectors with themselves and with one another we obtain for $\frac{1}{2}\sigma_3$ the eigenvalues $1, 0, 0, -1$, from which we can pick out the singlet and triplet states. We may express the reduction of the

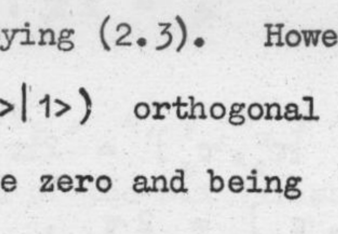

Fig. 4. The composition of two spin ½ vectors

product of the two doublet spin states symbolically as

$$D^{(2)} \otimes D^{(2)} = D^{(1)} \oplus D^{(3)} .$$

In elementary particle theory the SU_2 group is usually employed for isospin rather than for mechanical spin. The nucleon N can be in the two isospin $\frac{1}{2}$ states p denoted by $|\{2\},1\rangle$ or n denoted by $|\{2\},2\rangle$, so that the third component of isospin I_3 has the value $+\frac{1}{2}$ for a proton and $-\frac{1}{2}$ for a neutron. From these one can construct two-nucleon product states of total isospin 0 or 1. The mathematical discussion is just the same as before but it is customary to write the σ-operators as τ. Hence

$$\underset{\sim}{I} = \tfrac{1}{2}\underset{\sim}{\tau} ,$$

where

$$[\tau_2, \tau_3] = 2i\tau_1 \quad \text{etc.}$$

$$[\tau_3, \tau_+] = 2\tau_+ , \quad [\tau_3, \tau_-] = -2\tau_- , \quad [\tau_+, \tau_-] = \tau_3 . \tag{2.6}$$

These commutation relations are true for a system of two or more isofermions, i.e. isospin $\frac{1}{2}$ particles, because, for example,

$$[\tau_2(1) + \tau_2(2), \tau_3(1) + \tau_3(2)] = [\tau_2(1), \tau_3(1)] + [\tau_2(2), \tau_3(2)]$$
$$= i\,(\tau_3(1) + \tau_3(2)).$$

Consider the infinitesimal transformation for a single isofermion given by

$$U = 1 + i\,\varepsilon^A \tau_A .$$

The infinitesimals ε^1, ε^2, ε^3 are independent of each other and they can vary continuously. We say that the infinitesimal SU_2 is a <u>continuous group</u> of <u>order</u> 3. The infinitesimal transform of an isospinor f is

$$f' = Uf = (1 + i\varepsilon^A \tau_A) f$$

ao that

$$\delta f = f' - f = i\varepsilon^A \tau_A f .$$

If we take two successive transformations with

$$U_1 = 1 + i\varepsilon_1{}^A \tau_A , \qquad U_2 = 1 + i\varepsilon_2{}^A \tau_A$$

and put

$$U_2 U_1 = U_3 ,$$

we have

$$\varepsilon_3{}^A = \varepsilon_1{}^A + \varepsilon_2{}^A .$$

The $\varepsilon_3{}^A$'s are analytic functions of ε_1^1, ε_1^2, ε_1^3, ε_2^1, ε_2^2, ε_2^3; that is, the $\varepsilon_3{}^A$'s are continuous functions of the six variables and their partial derivatives with respect to the six variables exist.

We can now explain what is meant by a Lie group. Take a vector space of r dimensions with the coordinates $(x_1, x_2, x_3, \ldots x_r)$ and consider a set of transformations

$$x_i{}' = \phi_i\,(x_1, x_2, \ldots x_r;\, a_1, a_2, \ldots a_n) \qquad (2.7)$$

depending on the n independent, real and continuous parameters $a_1, a_2, \ldots, a_n$. Let ϕ_i be analytic functions of the a's and let zero values of the a's give the identity transformations

$$x_i = \phi_i\,(x_1, x_2, \ldots, x_r;\, 0, 0, \ldots, 0) .$$

Moreover let us suppose that there exists a set of values of the a's such that

$$x_i = \phi_i \, (x_1', x_2', \ldots, x_r'; \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n),$$

that is to say, the inverse of the transformations (2.7) exists. Furthermore consider the effect of successive transformations

$$x_i'' = \phi_i \, (x_1', x_2', \ldots, x_r'; a_1', a_2', \ldots a_n')$$
$$= \phi_i \{\phi_1 (x_1, x_2, \ldots, x_r; a_1, \ldots, a_n), \phi_2 (x_1, \ldots, x_r, a_1, \ldots, a_n), \ldots; \; a_1', a_2', \ldots, a_n'\}$$

and let us suppose that there exists a set of values of the a's such that

$$x_i'' = \phi_i (x_1, x_2, \ldots, x_r; a_1'', a_2'', \ldots, a_n'') .$$

The set of transformations then constitutes a group - a finite and continuous group. Clearly

$$a_s'' = \Psi_s \, (a_1, a_2, \ldots, a_n; a_1', a_2', \ldots a_n')$$

and, if Ψ_s is an analytic function of the a's and the a''s, we have a Lie group of order n. The infinitesimal SU_2 is an elementary example of a Lie group of order 3. By iterating we deduce that the finite SU_2 is a Lie group of order 3.

Let us make an infinitesimal transformation of a function f of the x's in the neighbourhood of $a_1 = a_2 = \ldots = a_n = 0$, so that

$$\delta f = \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial a_\mu} \delta a_\mu = \delta a_\mu \left. \frac{\partial \phi_i}{\partial a_\mu} \right|_{a_1 = a_2 = \ldots = a_n = 0} \frac{\partial f}{\partial x_i} ,$$

which we write

$$\delta f = \varepsilon^A L_A f ,$$

where ε^A stands for δa_μ and

$$L_A = \left. \frac{\partial \phi_i}{\partial a_\mu} \right|_{a_1 = a_2 = \ldots = a_n = 0} \frac{\partial}{\partial x_i} .$$

It may then be shown[5] that these n L_A's satisfy the commutation

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(5) M. Hamermesh, Group Theory and Its Application to Physical Problems. pp. 299 et seqq. (Addison Wesley, 1962).

relations

$$[L_A, L_B] = C_{AB}{}^D L_D \qquad (2.8)$$

summed over D. We notice the resemblance between this equation and (2.6).

CHAPTER III:

Lie Algebras

We turn our attention to the algebra of Lie groups. We define the product of two operators O_1 and O_2 as the commutator $[O_1, O_2]$. We then say that a set of operators forms a Lie algebra, if every linear combination of them belongs to the set and if the product of any two of them belongs to the set. The order of the set is the number of linearly independent operators. The algebra is Abelian, if all the operators commute with each other.

Take a set of independent L_A's that constitute a Lie algebra. The two defining properties give

$$[L_A, L_B] = C_{AB}{}^{D} L_D , \tag{3.1}$$

which is just (2.8). It follows that

$$C_{AB}{}^{D} = - C_{BA}{}^{D}, \tag{3.2}$$

so that $C_{AB}{}^{D}$ vanishes if L_A and L_B commute. Jacobi's identity

$$\sum_{ABD} [L_A, [L_B, L_D]] = 0$$

yields

$$\sum_{ABD} C_{BD}{}^{E} C_{AE}{}^{F} L_F = 0 ,$$

and so

$$C_{AB}{}^{E} C_{ED}{}^{F} + C_{BD}{}^{E} C_{EA}{}^{F} + C_{DA}{}^{E} C_{EB}{}^{F} = 0 \tag{3.3}$$

because the L_F's are independent.

It is usual to treat the Lie algebra by a matrix representation. This signifies that we take square matrices obeying the same algebraic relations as the L's . If the matrices have ℓ rows and ℓ columns, we have an ℓ-dimensional representation. The integer ℓ can assume an infinity of values. When we write $L_{As}{}^{t}$, we shall mean the st-element of L_A, s labelling the rows, t labelling the columns and L_A denoting the matrix representative of the Ath operator. We may show that (3.2) and (3.3) are sufficient to establish the existence of matrices satisfying (3.1); in fact a possible choice of these matrices is the set of structure constants $C_{AB}{}^{D}$ themselves with a minus sign because

$$(-C_A \cdot -C_B - -C_B \cdot -C_A)_D{}^F - C_{AB}{}^E(-C_E)_D{}^F$$

$$= C_{AD}{}^E C_{BE}{}^F - C_{BD}{}^S C_{AS}{}^F + C_{AB}{}^E C_{ED}{}^F$$

$$= C_{AB}{}^E C_{ED}{}^F + C_{BD}{}^E C_{EA}{}^F + C_{DA}{}^E C_{EB}{}^F = 0 .$$

When the matrices are chosen in this way, we are said to have the regular representation of the group. The dimension of the regular representation is the order of the ~~group~~ set, since the labels of the rows and columns are just the subscripts of the L_A's.

Next we introduce the notion of a subalgebra, by which is meant a subset of the matrices of an algebra which is itself an algebra. In the case of a Lie algebra a subalgebra would be constituted by a subset of matrices obeying (3.1), the L_A, L_B, L_D all belonging to the subset. An invariant subalgebra is a subalgebra such that the product of A and B belongs to the subalgebra if A does, while it is sufficient that B belong to the algebra only. Thus for an invariant Lie subalgebra $[A, L_F]$ belong to the subalgebra if A does. A simple algebra is one that has no invariant subalgebra.

A <u>semi-simple</u> algebra is one that has no Abelian invariant subalgebra. <u>Cartan's criterion</u> states that a necessary and sufficient condition for a Lie algebra to be semi-simple is that the matrix g_{AB} defined by

$$g_{AB} = C_{AD}{}^{E} C_{BE}{}^{D}$$

be non-singular; that is to say, the determinant of the g_{AB}'s is not zero. We may note that

$$g_{AB} = (C_A)_D{}^E (C_B)_E{}^D = (C_A C_B)_D{}^D = \mathrm{tr}\, C_A C_B ,$$

the trace of the product of C_A and C_B.

Let us prove that g_{AB} is singular, if it has an Abelian invariant subalgebra. If the order of the subset is r, let us suppose that the matrices of the algebra are so arranged that the elements of the subalgebra are independent linear combinations of the first r matrices. We shall employ small letters to refer to these matrices. L_a and L_b commute, so $C_{ab}{}^{D}$ vanishes. Moreover, since $[L_A, L_b]$ belongs to the subgroup, it follows that in the summation $C_{Ab}{}^{B} L_B$ we get no contribution when $B > r$; that is to say,

$$C_{Ab}{}^{B} = 0 \quad \text{for} \quad B > r .$$

Hence

$$g_{Ab} = C_{AD}{}^{E} C_{bE}{}^{D} = C_{Ad}{}^{E} C_{bE}{}^{d} = C_{Ad}{}^{e} C_{be}{}^{d} = 0 ,$$

which shows that all the elements of the b^{th} column vanish so that the matrix g_{AB} has vanishing determinant. We shall assume the converse: if g_{AB} has vanishing determinant, the algebra has an Abelian invariant sub-algebra. The proof of this requires a rather lengthy study of the theory of continuous groups[6], through which there is no legitimate short-cut.

Suppose that we have a Lie algebra of order m ; that is, we have n independent L_A's . Of these m , say, will commute among themselves. We call m the <u>rank</u> of the group. It is customary to denote the commuting L_A's by H_i $(i = 1, 2, \ldots, m)$ and the remaining ones by E_α $(\alpha = n-m, n-m+1, \ldots, n)$. We may use another Latin letter for i and another Greek letter for α . The rank of SU_2 is one because no two of the τ_1, τ_2, τ_3 commute. The groups in which we shall be interested and to which we shall confine our calculations are all of rank 2 . The H_1 and H_2 are proportional to operators representing some physical quantity like charge, hypercharge, strangeness, the third component of isospin. Since H_1 , H_2 are observables, their matrix representations are hermitian.

Our next objective is to express the commutation relations (3.1) in a convenient form. To do this we turn our attention to the structure constants, since it they that express the properties of the Lie group. We work in the regular representation and identify $C_{iA}^{\ \ B}$ with $-H_{iA}^{\ \ B}$. By making a similarity transformation

$$C_D' = S^{-1} C_D S ,$$

where S is independent of D , we construct a representation in which

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(6) Cf. E. Cartan, <u>Sur la Structure des Groupes de Transformations Finis et Continus</u> (Thèse, Paris, 1894) reprinted in E. Cartan, <u>Oeuvres Complètes</u>, Partie I, Vol. I (Gauthier-Villars, Paris, 1952); L. P. Eisenhart, Continuous Groups of Transformations, Chap. I-IV (Princeton Univ. Press, 1933); G. Racah, <u>Group Theory and Spectroscopy</u>, CERN Report 61-8, pp. 1-56.

C_1' and C_2' are real and diagonal. Let us suppress the primes on the understanding that we are working in this representation. Then

$$[H_i, E_\alpha] = C_{i\alpha}{}^A L_A = C_{i\alpha}{}^\alpha L_\alpha = r_i(\alpha) E_\alpha ,$$

where we have written $r_i(\alpha)$ for $C_{i\alpha}{}^\alpha$, not summed over α. Since $[H_i, H_j]$ vanishes, so does $C_{ij}{}^A$. We may collect our results for these structure constants by writing

$$C_{ij}{}^A = 0 , \quad C_{i\alpha}{}^j = 0 , \quad C_{i\alpha}{}^\beta = r_i(\alpha)\, \delta_\alpha{}^\beta \qquad (3.4)$$

and we recall that i , j have the values 1 and 2 , α , β have the values 3 to n , and A has the values 1 to n . We shall denote the pair of numbers $(r_1(\alpha), r_2(\alpha))$ by $\underline{r}(\alpha)$, and this we call a <u>root vector</u>.

We study some properties of a semi-simple Lie algebra, that is, one which has no Abelian invariant sub-algebra. Since

$$[H_i, [E_\alpha, E_\beta]] = C_{\alpha\beta}{}^A [H_i, L_A] = C_{\alpha\beta}{}^A C_{iA}{}^B L_B ,$$

$$0 = \Sigma [H_i, [E_\alpha, E_\beta]] = (C_{\alpha\beta}{}^A C_{iA}{}^B + C_{\beta i}{}^A C_{\alpha A}{}^B + C_{i\alpha}{}^A C_{\beta A}{}^B) L_B ,$$

(3.4) gives

$$0 = C_{\alpha\beta}{}^\gamma C_{i\gamma}{}^B - r_i(\beta)\, C_{\alpha\beta}{}^B + r_i(\alpha)\, C_{\beta\alpha}{}^B ,$$

$$(r_i(\alpha) + r_i(\beta))\, C_{\alpha\beta}{}^B = C_{\alpha\beta}{}^\gamma C_{i\gamma}{}^B .$$

We take successively B equal to k (1 or 2) and B equal to δ (> 2) .

$$(r_i(\alpha) + r_i(\beta))\, C_{\alpha\beta}{}^k = C_{\alpha\beta}{}^\gamma C_{i\gamma}{}^k = 0 \qquad (3.5)$$

by (3.4),

$$(r_i(\alpha) + r_i(\beta))\, C_{\alpha\beta}{}^\delta = C_{\alpha\beta}{}^\gamma C_{i\gamma}{}^\delta = C_{\alpha\beta}{}^\delta\, r_i(\delta)$$

that is,

$$(r_i(\alpha) + r_i(\beta) - r_i(\delta))\, C_{\alpha\beta}{}^{\delta} = 0 \,. \qquad (3.6)$$

To discuss (3.5) we assume that corresponding to every root $\underline{r}(\alpha)$ there exists another which is just its negative $-\underline{r}(\alpha)$. We write $-\alpha$ for the value of β that gives this root, so that $\underline{r}(-\alpha)$ is just $-\underline{r}(\alpha)$. For this value of β we write $C_{\alpha\beta}{}^{k}$ as $r^k(\alpha)$ and for other values of β equation (3.5) shows that $C_{\alpha\beta}{}^{k}$ vanishes. Thus we write

$$C_{\alpha,-\alpha}{}^{k} = r^k(\alpha) \; ; \qquad C_{\alpha\beta}{}^{k} = 0 \text{ for } \beta \neq -\alpha \,. \quad (3.7)$$

Similarly equation (3.6) shows that $C_{\alpha\beta}{}^{\delta}$ vanishes unless $\underline{r}(\delta) = \underline{r}(\alpha) + \underline{r}(\beta)$. When this condition is satisfied, we agree to write $C_{\alpha\beta}{}^{\delta}$ as $N_{\alpha\beta}$ and so

$$C_{\alpha\beta}{}^{\alpha+\beta} = N_{\alpha\beta} = -N_{\beta\alpha} \; ; \quad C_{\alpha\beta}{}^{\delta} = 0 \text{ for } \underline{r}(\delta) \neq \underline{r}(\alpha) + \underline{r}(\beta) \,. \qquad (3.8)$$

There is no summation over the repeated α indices in (3.7) or (3.8). If we understand by E_α the E with root $\underline{r}(\alpha)$ and by $E_{-\alpha}$ the E with root $-\underline{r}(\alpha)$, we deduce that

$$[E_\alpha, E_{-\alpha}] = C_{\alpha,-\alpha}{}^{D} L_D = C_{\alpha,-\alpha}{}^{i} H_i + C_{\alpha,-\alpha}{}^{\delta} E_\delta = r^i(\alpha) H_i \qquad (3.9)$$

by (3.8), and that for $\beta \neq -\alpha$

$$[E_\alpha, E_\beta] = C_{\alpha\beta}{}^{D} L_D = C_{\alpha\beta}{}^{i} H_i + C_{\alpha\beta}{}^{\delta} E_\delta = N_{\alpha\beta}\, E_{\alpha+\beta} \qquad (3.10)$$

by (3.7) and (3.8).

Let us check that in a semi-simple algebra to every root $\underline{r}(\alpha)$ there corresponds another $-\underline{r}(\alpha)$. By definition

$$g_{\alpha\sigma} = C_{\alpha D}{}^{E} C_{\sigma E}{}^{D} = C_{\alpha i}{}^{E} C_{\sigma E}{}^{i} + C_{\alpha\beta}{}^{E} C_{\sigma E}{}^{\beta}$$
$$= C_{\alpha i}{}^{\gamma} C_{\sigma\gamma}{}^{i} + C_{\alpha,-\alpha}{}^{k} C_{\sigma k}{}^{-\alpha} + \sum_{\beta \neq -\alpha} C_{\alpha\beta}{}^{\alpha+\beta} C_{\sigma,\alpha+\beta}{}^{\beta} \tag{3.11}$$

by (3.4), (3.7) and (3.8). When $\sigma \neq -\alpha$, each term on the right hand side vanishes and so does $g_{\alpha\sigma}$. This means that, if there is no root $-\underline{r}(\alpha)$, the matrix g_{AB} is singular. Then by Cartan's criterion the algebra has an Abelian invariant subalgebra, which is contrary to supposition.

The next problem is to find a convenient form for the matrix g_{AB}. We first note that

$$g_{k\alpha} = g_{\alpha k} = C_{\alpha D}{}^{E} C_{kE}{}^{D} = C_{\alpha i}{}^{E} C_{kE}{}^{i} + C_{\alpha\beta}{}^{E} C_{kE}{}^{\beta}$$
$$= C_{\alpha i}{}^{\gamma} C_{k\gamma}{}^{i} + C_{\alpha\beta}{}^{\gamma} C_{k\gamma}{}^{\beta} = 0$$

by (3.4). We have to consider only g_{ij}, and $g_{\alpha\sigma}$ with $\sigma = -\alpha$. Of course $g_{\alpha,-\alpha}$ cannot vanish for, if it did, $g_{\alpha B}$ would vanish and the matrix g_{AB} would be singular. Let us now multiply each E_{α} by a real number ℓ_{α} leaving H_i unchanged. From the equation $[H_i, E_{\alpha}] = r_i(\alpha) E_{\alpha}$ we deduce that $r_i(\alpha)$, i.e. $C_{i\alpha}{}^{\alpha}$, is unchanged. From $[E_{\alpha}, E_{-\alpha}] = r^i(\alpha) H_i$ we deduce that

$$C_{\alpha,-\alpha}{}^{i} = r^i(\alpha) \rightarrow \ell_{\alpha} \ell_{-\alpha} r^i(\alpha) = \ell_{\alpha} \ell_{-\alpha} C_{\alpha,-\alpha}{}^{i}$$

and from $[E_{\alpha}, E_{\beta}] = C_{\alpha\beta}{}^{\alpha+\beta} E_{\alpha+\beta}$ we deduce that

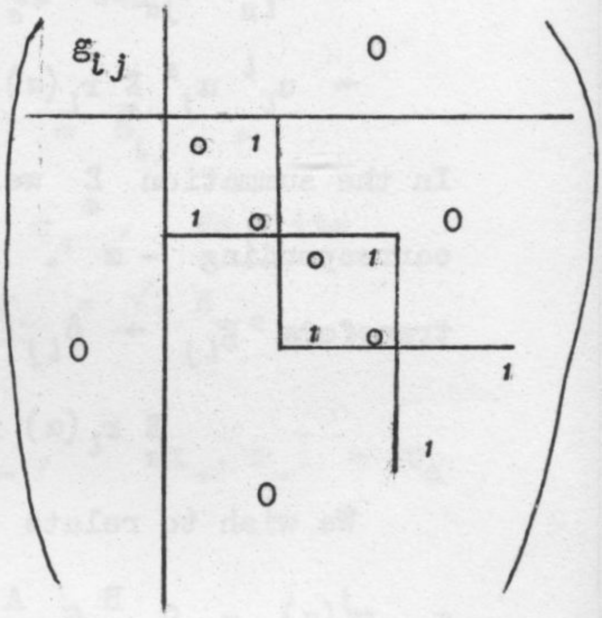
$$C_{\alpha\beta}{}^{\alpha+\beta} \rightarrow \frac{\ell_{\alpha} \ell_{\beta}}{\ell_{\alpha+\beta}} C_{\alpha\beta}{}^{\alpha+\beta}$$

Now by (3.11)

$$g_{-\alpha,\alpha} = g_{\alpha,-\alpha} = C_{\alpha i}{}^{\gamma} C_{-\alpha\gamma}{}^{i} + C_{\alpha,-\alpha}{}^{k} C_{-\alpha k}{}^{-\alpha} + \sum_{\beta \neq -\alpha} C_{\alpha\beta}{}^{\alpha+\beta} C_{-\alpha,\alpha+\beta}{}^{\beta} \tag{3.12}$$

and we see that each term on the right hand side becomes multiplied by the factor $\ell_\alpha \ell_{-\alpha}$. If we choose our ℓ's such that

$$\ell_\alpha \ell_{-\alpha} = \frac{1}{g_{\alpha,-\alpha}},$$



we obtain

$$g_{-\alpha,\alpha} = g_{\alpha,-\alpha} = 1,$$

and the matrix g_{AB} is as shown with g_{ij} non-singular because g_{AB} is non-singular.

Lastly we want to show that we may express g_{ij} as δ_{ij} by suitably transforming the H's and leaving the E's unaltered. We make the transformation $H_i \to u_i^{\ j} H_j$ and the relation $[H_i, E_\alpha] = C_{i\alpha}^{\ \ \alpha} E_\alpha$ shows that

$$r_i(\alpha) = C_{i\alpha}^{\ \ \alpha} \to u_i^{\ j} C_{j\alpha}^{\ \ \alpha} = u_i^{\ j} r_j(\alpha).$$

Also we deduce from $[E_\alpha, E_{-\alpha}] = C_{\alpha,-\alpha}^{\ \ \ \ i} H_i = r^i(\alpha) H_i$ that

$$r^i(\alpha) = C_{\alpha,-\alpha}^{\ \ \ \ i} \to (u^{-1})_j^{\ i} C_{\alpha,-\alpha}^{\ \ \ \ j} = (u^{-1})_j^{\ i} r^j(\alpha).$$

Equation (3.12) gives

$$g_{\alpha,-\alpha} = C_{\alpha i}^{\ \ \alpha} C_{-\alpha,\alpha}^{\ \ \ \ i} + C_{\alpha,-\alpha}^{\ \ \ \ i} C_{-\alpha i}^{\ \ \ -\alpha} + \sum_{\beta\neq-\alpha} C_{\alpha\beta}^{\ \ \alpha+\beta} C_{-\alpha,\alpha+\beta}^{\ \ \ \ \beta}$$

not summed over the α's and so

$$g_{\alpha,-\alpha} \to u_i^{\ j} C_{\alpha j}^{\ \ \alpha} (u^{-1})_m^{\ i} C_{-\alpha,\alpha}^{\ \ \ \ m} + C_{\alpha,-\alpha}^{\ \ \ \ j} (u^{-1})_j^{\ i} C_{-\alpha t}^{\ \ \ -\alpha} u_i^{\ t} +$$

$$\sum_{\beta\neq\alpha} C_{\alpha\beta}^{\ \ \alpha+\beta} C_{-\alpha,\alpha+\beta}^{\ \ \ \ \beta} = g_{\alpha,-\alpha}.$$

Besides

$$g_{ij} = C_{iA}{}^{B} C_{jB}{}^{A} = C_{im}{}^{B} C_{jB}{}^{m} C_{i\mu}{}^{B} C_{jB}{}^{\mu}$$

$$= C_{i\mu}{}^{\alpha} C_{j\alpha}{}^{\mu} = \sum_{\alpha} r_i(\alpha)\, r_j(\alpha)\,, \text{ by } (3.4)$$

$$\rightarrow u_i{}^{l} u_j{}^{s} \sum_{\alpha} r_l(\alpha)\, r_s(\alpha) = u_i{}^{l} u_j{}^{s} g_{ls}\,.$$

In the summation $\sum_{\alpha}$ we agree to sum over each α but not over its corresponding $-\alpha$. By suitably choosing the u's we can transform $g_{ij} \rightarrow \delta_{ij}$. This also gives

$$\sum_{\alpha} r_i(\alpha)\, r_j(\alpha) = \delta_{ij}\,.$$

We wish to relate $r^{i}(\alpha)$ to $r_i(\alpha)$.

$$g_{ij}\, r^{j}(\alpha) = C_{iA}{}^{B} C_{jB}{}^{A} C_{\alpha,-\alpha}{}^{j} = C_{iA}{}^{B} C_{\alpha,-\alpha}{}^{D} C_{DB}{}^{A}\,,$$

because $C_{\alpha,-\alpha}$ with $D = \beta$ gives no contribution. Employing (3.3) we deduce

$$g_{ij}\, r^{j}(\alpha) = -C_{iA}{}^{B}\left(C_{-\alpha B}{}^{D} C_{D\alpha}{}^{A} + C_{B\alpha}{}^{D} C_{D,-\alpha}{}^{A}\right)$$

$$= -C_{iA}{}^{B} C_{D,-\alpha}{}^{A} C_{B\alpha}{}^{D} + C_{D\alpha}{}^{B} C_{iB}{}^{A} C_{A,-\alpha}\,,$$

where A and B have been interchanged

$$= -C_{iA}{}^{B} C_{D,-\alpha}{}^{A} C_{B\alpha}{}^{D} - C_{D\alpha}{}^{B}\left(C_{B,-\alpha}{}^{A} C_{Ai}{}^{D} + C_{-\alpha i}{}^{A} C_{AB}{}^{D}\right)$$

$$= -C_{D\alpha}{}^{B} C_{-\alpha i}{}^{A} C_{AB}{}^{D}\,,$$ where B and D had been interchanged in the first term. Then by (3.4)

$$g_{ij}\, r^{j}(\alpha) = C_{-\alpha i}{}^{-\alpha} C_{-\alpha B}{}^{D} C_{\alpha D}{}^{B} = -g_{-\alpha,\alpha}\, r_i(-\alpha) = r_i(\alpha)$$

so that with our g_{ij}

$$r^{i}(\alpha) = r_i(\alpha)\,.$$

We conclude by re-writing the commutation relations

$$[H_l, H_j] = 0, \qquad [H_l, E_\alpha] = r_l(\alpha) E_\alpha \tag{3.13}$$
$$[E_\alpha, E_{-\alpha}] = r^l(\alpha) H_l, \qquad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$$

where

$$r^l(\alpha) = r_l(\alpha), \qquad \sum_\alpha r_l(\alpha) r_j(\alpha) = \delta_{lj} .$$

For the SU_2 group the only diagonal matrix is τ_3 . We write

$$\tau_3 = 2 H_l , \quad \tau_+ = \sqrt{2} E_\alpha , \quad \tau_- = \sqrt{2} E_{-\alpha}$$

and the relations

$$[\tau_3, \tau_+] = 2\tau_+ , \qquad [\tau_3, \tau_-] = -2\tau_- , \qquad [\tau_+, \tau_-] = \tau_3$$

are just

$$[H_l, E_\alpha] = E_\alpha , \qquad [H_l, E_{-\alpha}] = -E_{-\alpha}$$
$$[E_\alpha, E_{-\alpha}] = H_l$$

in agreement with (3.13).

CHAPTER IV:

Root Diagrams

We employ equations (3.13) to study some general properties of the root vectors. From the Jacobi identity

$$[E_\alpha, [E_\beta, E_{-\beta}]] + [E_\beta, [E_{-\beta}, E_\alpha]] + [E_{-\beta}, [E_\alpha, E_\beta]] = 0$$

we deduce

$$\begin{aligned} 0 &= r^l(\beta)\,[E_\alpha, H_l] + N_{-\beta,\alpha}[E_\beta, E_{\alpha-\beta}] + N_{\alpha\beta}[E_{-\beta}, E_{\alpha+\beta}] \\ &= -r^l(\beta)\, r_l(\alpha)\, E_\alpha + N_{-\beta,\alpha} N_{\beta,\alpha-\beta} E_\alpha + N_{\alpha\beta} N_{-\beta,\alpha+\beta} E_\alpha \\ &= \{-(\underset{\sim}{r}(\alpha)\cdot\underset{\sim}{r}(\beta)) - N_{\alpha-\beta,\beta} N_{-\beta,\alpha} + N_{\alpha\beta} N_{-\beta,\alpha+\beta}\}\, E_\alpha , \end{aligned}$$

so that

$$N_{\alpha\beta} N_{-\beta,\alpha+\beta} = N_{\alpha-\beta,\beta} N_{-\beta,\alpha} + (\underset{\sim}{r}(\alpha)\cdot\underset{\sim}{r}(\beta)) .$$

In this equation we replace $\underset{\sim}{r}(\alpha)$ by $\underset{\sim}{r}(\alpha) + s\,\underset{\sim}{r}(\beta)$, where s is an integer. Then α is replaced by $\alpha + s\beta$ in the suffixes and we obtain

$$N_{\alpha+s\beta,\beta} N_{-\beta,\alpha+(s+1)\beta} = N_{\alpha+(s-1)\beta,\beta} N_{-\beta,\alpha+s\beta} + (\underset{\sim}{r}(\alpha) + s\,\underset{\sim}{r}(\beta)\cdot\underset{\sim}{r}(\beta)).$$

We express this as

$$\mu_s = \mu_{s-1} + (\underset{\sim}{r}(\alpha)\cdot\underset{\sim}{r}(\beta)) + s\,|\underset{\sim}{r}(\beta)|^2 \qquad (4.1)$$

with

$$\mu_s = N_{\alpha+s\beta,\beta} N_{-\beta,\alpha+(s+1)\beta} . \qquad (4.2)$$

Consider the sequences

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta} \; ; \qquad [E_{\alpha+\beta}, E_\beta] = N_{\alpha+\beta,\beta}\, E_{\alpha+2\beta} \; ; \; \ldots$$
$$[E_\alpha, E_{-\beta}] = N_{\alpha,-\beta}\, E_{\alpha-\beta} \; ; \qquad [E_{\alpha-\beta}, E_{-\beta}] = N_{\alpha-\beta,-\beta}\, E_{\alpha-2\beta} \; ; \; \ldots .$$

Since there is a finite number of $C_{i\alpha}^{\ \alpha}$'s, there is a finite number of roots. Suppose that we can go up from $\mathfrak{r}(\alpha)$ to $\mathfrak{r}(\alpha) + k\,\mathfrak{r}(\beta)$ and down from $\mathfrak{r}(\alpha)$ to $\mathfrak{r}(\alpha) - k'\,\mathfrak{r}(\beta)$. We call these a <u>string of roots</u>. Then $\mu_k = \mu_{-k'-1} = 0$. In (4.1) we allow s to take values down to $-k'$.

$$\mu_s = \mu_{s-1} + (\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta)) + s\,|\mathfrak{r}(\beta)|^2$$
$$\mu_{s-1} = \mu_{s-2} + (\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta)) + (s-1)\,|\mathfrak{r}(\beta)|^2$$
$$\ldots$$
$$\mu_{-k'} = 0 + (\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta)) - k'\,|\mathfrak{r}(\beta)|^2 .$$

On addition we find that

$$\mu_s = (s + k' + 1)\,(\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta)) + \tfrac{1}{2}\,(s + k' + 1)\,(s - k')\,|\mathfrak{r}(\beta)|^2 \tag{4.3}$$

If we put $s = k$, we deduce that

$$0 = (k + k' + 1)\,\{(\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta)) + \tfrac{1}{2}\,(k - k')\,|r(\beta)|^2\} ,$$

which shows that

$$\frac{2\,(\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta))}{|\mathfrak{r}(\beta)|^2} = k' - k , \tag{4.4}$$

an integer. Moreover

$$\mathfrak{r}(\alpha) - \frac{2\,(\mathfrak{r}(\alpha) \cdot \mathfrak{r}(\beta))\,\mathfrak{r}(\beta)}{|\mathfrak{r}(\beta)|^2} = \mathfrak{r}(\alpha) + (k - k')\,\mathfrak{r}(\beta) \tag{4.5}$$

is a root because it lies on the string from $r(\alpha) - k' r(\beta)$ to $r(\alpha) + k\, r(\beta)$.

This last result admits of a simple geometrical interpretation. If θ is the angle between $r(\alpha)$ and $r(\beta)$,

$$\frac{(r(\alpha) \cdot r(\beta))}{|r(\alpha)| \cdot |r(\beta)|} = \cos\theta .$$

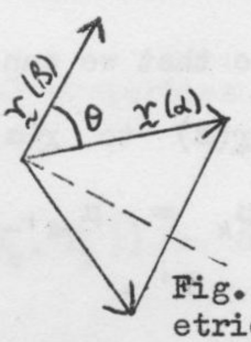


Fig. 5: The geometrical interpretation of equation (4.5).

When $r(\alpha)$ is reflected in the line through the origin perpendicular to $r(\beta)$, we obtain

$$r(\alpha) - 2\, r(\alpha) \cos\theta \frac{r(\beta)}{|r(\beta)|} = r(\alpha) - \frac{2(r(\alpha) \cdot r(\beta))\, r(\beta)}{|r(\beta)|^2} ,$$

which is just the root in (4.5).

So far the only symmetry property of $N_{\alpha\beta}$ that we have noted is $N_{\beta\alpha} = - N_{\alpha\beta}$. Take three root vectors $r(\alpha)$, $r(\beta)$, $r(\gamma)$ satisfying $r(\alpha) + r(\beta) + r(\gamma) = 0$. Then $r(\alpha) + r(\beta) = - r(\gamma)$ is a root and so are $r(\beta) + r(\gamma)$, $r(\gamma) + r(\alpha)$. Hence $N_{\beta\gamma}$, $N_{\gamma\alpha}$, $N_{\alpha\beta}$ exist. Now

$$\begin{aligned}
0 &= [E_\alpha, [E_\beta, E_\gamma]] + [E_\beta, [E_\gamma, E_\alpha]] + [E_\gamma, [E_\alpha, E_\beta]] \\
&= N_{\beta\gamma} [E_\alpha, E_{-\alpha}] + N_{\gamma\alpha} [E_\beta, E_{-\beta}] + N_{\alpha\beta} [E_\gamma, E_{-\gamma}] \\
&= \{ r^l(\alpha) N_{\beta\gamma} + r^l(\beta) N_{\gamma\alpha} + r^l(\gamma) N_{\alpha\beta} \} H_l \\
&= \{ r_l(\alpha) N_{\beta,-\alpha-\beta} + r_l(\beta) N_{-\alpha-\beta,\alpha} - [r_l(\alpha) + r_l(\beta)] N_{\alpha\beta} \} H_l \\
&= r_l(\alpha) H_l (N_{\beta,-\alpha-\beta} - N_{\alpha\beta}) + r_l(\beta) H_l (N_{-\alpha-\beta,\alpha} - N_{\alpha\beta}) .
\end{aligned}$$

Since the choice of $r(\alpha)$ and $r(\beta)$ is at our disposal, the quantities in the brackets must vanish separately and thus

$$N_{\alpha\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} \quad . \qquad (4.6)$$

When $N_{\alpha\beta}$ exists, so does $N_{-\alpha,-\beta}$. When putting the matrix g_{AB} into standard form we made the transformations $E_\alpha \to \ell_\alpha E_\alpha$ and these gave

$$N_{\alpha\beta} = C_{\alpha\beta}{}^{\alpha+\beta} \to \frac{\ell_\alpha \ell_\beta}{\ell_{\alpha+\beta}} C_{\alpha\beta}{}^{\alpha+\beta} = \frac{\ell_\alpha \ell_\beta}{\ell_{\alpha+\beta}} N_{\alpha\beta} ,$$

so that

$$\frac{N_{\alpha\beta}}{N_{-\alpha,-\beta}} \to \frac{\ell_\alpha \ell_\beta \ell_{-\alpha-\beta}}{\ell_{\alpha+\beta} \ell_{-\alpha} \ell_{-\beta}} \frac{N_{\alpha\beta}}{N_{-\alpha,-\beta}} = \frac{\ell_\alpha}{\ell_{-\alpha}} \frac{\ell_\beta}{\ell_{-\beta}} \frac{\ell_{-\alpha-\beta}}{\ell_{\alpha+\beta}} \frac{N_{\alpha\beta}}{N_{-\alpha,-\beta}} .$$

We have already fixed $\ell_\alpha \ell_{-\alpha}$ but not $\dfrac{\ell_\alpha}{\ell_{-\alpha}}$ and this fraction we choose in such a way that the last expression becomes -1 . Then

$$N_{\alpha\beta} = - N_{-\alpha,-\beta} \quad . \qquad (4.7)$$

We are now in a position to say something about the numerical value of $N_{\alpha\beta}$. From (4.2) and (4.3) we deduce that

$$\begin{aligned} N_{\alpha\beta} N_{-\beta,\alpha+\beta} &= \mu_0 = (k'+1)\,(\mathfrak{r}(\alpha)\cdot\mathfrak{r}(\beta)) - \tfrac{1}{2}k'(k'+1)\,|\mathfrak{r}(\beta)|^2 \\ &= -\tfrac{1}{2}k\,(k'+1)\,|\mathfrak{r}(\beta)|^2 , \text{ by } (4.4) . \end{aligned}$$

Then (4.6) and (4.7) give

$$\tfrac{1}{2}k\,(k'+1)\,|\mathfrak{r}(\beta)|^2 = - N_{\alpha\beta} N_{-\beta,\alpha+\beta} = - N_{\alpha\beta} N_{-\alpha,-\beta} = (N_{\alpha\beta})^2 ,$$

that is,

$$N_{\alpha\beta} = \pm\sqrt{\{\tfrac{1}{2}k\,(k'+1)\}}\,|\mathfrak{r}(\beta)| . \qquad (4.8)$$

The result in (4.4) that $\dfrac{2\,(\mathfrak{r}(\alpha)\cdot\mathfrak{r}(\beta))}{|\mathfrak{r}(\beta)|^2}$, and similarly $\dfrac{2(\mathfrak{r}(\alpha)\cdot\mathfrak{r}(\beta))}{|\mathfrak{r}(\alpha)|^2}$, are integers allows only particular values for the angle θ between $\mathfrak{r}(\alpha)$ and $\mathfrak{r}(\beta)$. Writing the integers m and n we have

$$(\underset{\sim}{r}(\alpha)\cdot\underset{\sim}{r}(\beta)) = \frac{m}{2}|\underset{\sim}{r}(\beta)|^2 = \frac{n}{2}|\underset{\sim}{r}(\alpha)|^2 = \pm\sqrt{\left(\frac{mn}{4}\right)}\,|\underset{\sim}{r}(\alpha)|\cdot|\underset{\sim}{r}(\beta)|$$

$$\cos^2\theta = \frac{1}{4}mn .$$

Hence we have the following allowed values:

mn	0	1	2	3	4
θ	$\frac{\pi}{2}$	$\frac{\pi}{3},\frac{2\pi}{3}$	$\frac{\pi}{4},\frac{3\pi}{4}$	$\frac{\pi}{6},\frac{5\pi}{6}$	$0, \pi$
$\frac{\lvert\underset{\sim}{r}(\alpha)\rvert}{\lvert\underset{\sim}{r}(\beta)\rvert}=\sqrt{\frac{m}{n}}$	0	1	$\sqrt{2}$	$\sqrt{3}$	1 or 2 .

Since for every root $\underset{\sim}{r}(\alpha)$ there exists $-\underset{\sim}{r}(\alpha)$, the supplementary angles $\pi-\theta$ tell us nothing new. In writing zero for $\frac{|\underset{\sim}{r}(\alpha)|}{|\underset{\sim}{r}(\beta)|}$ in the case of $mn = 0$ we have disregarded the trivial case where both $\underset{\sim}{r}(\alpha)$ and $\underset{\sim}{r}(\beta)$ vanish . The cases of $\theta = 0$ and $\theta = \frac{\pi}{2}$ give only roots lying along one straight line and are therefore degenerate cases of Lie groups of rank 2 that do not interest us here. When putting down diagrams for the roots we must satisfy the normalization condition $\sum_{\alpha} r_i(\alpha)\, r_i(\alpha) = 1$, the summations being performed only over the $\underset{\sim}{r}(\alpha)$'s in a half-plane. Taking successively $\theta = \pi/3,\ \pi/4,\ \pi/6$ we draw the root diagrams. They correspond, respectively, to the unimodular unitary group in three dimensions SU_3 , the five-dimensional orthogonal group B_2, and the exceptional group G_2 . We notice how we can go from one root to another by reflection.

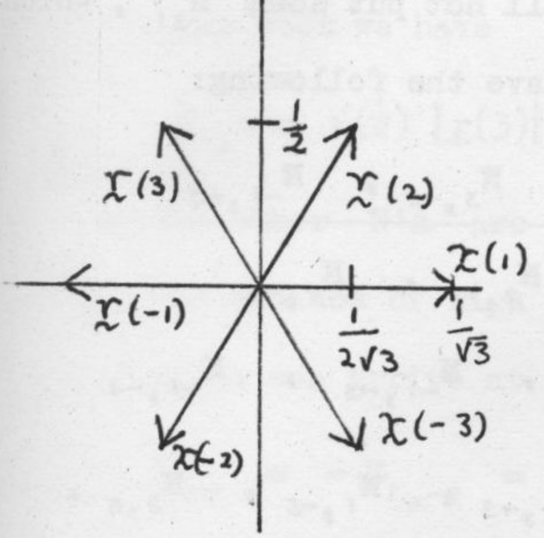


Fig. 6: The root diagram for SU_3.

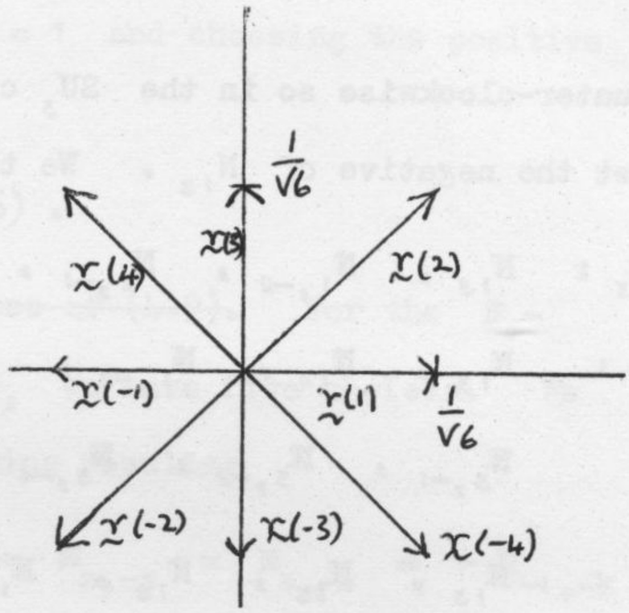


Fig. 7: The root diagram for B_2.

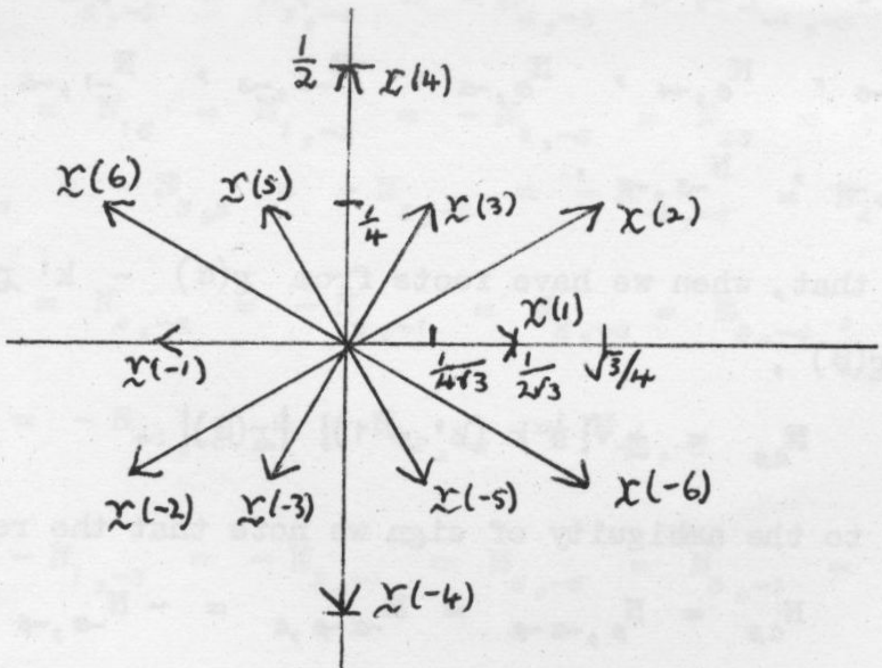


Fig. 8: The root diagram for G_2.

The root diagrams provide some information about $N_{\alpha\beta}$. This exists only when $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ is a root, so for the SU_3 group N_{12} does not exist but N_{13} does. In listing the N's we agree to go counter-clockwise so in the SU_3 case we shall not put down N_{31}, which is just the negative of N_{13}. We therefore have the following:

SU_3 : N_{13}, $N_{1,-2}$, $N_{2,-1}$, $N_{2,-3}$, $N_{3,-2}$, $N_{-1,-3}$

B_2 : N_{13}, N_{14}, $N_{1,-2}$, $N_{1,-3}$, $N_{2,-1}$, $N_{2,-3}$

$N_{3,-1}$, $N_{3,-2}$, $N_{3,-4}$, $N_{4,-3}$, $N_{-1,-3}$, $N_{-1,-4}$

G_2 : N_{13}, N_{15}, N_{16}, $N_{1,-2}$, $N_{1,-3}$, $N_{1,-5}$, $N_{2,6}$,

$N_{2,-3}$, $N_{2,-4}$, $N_{3,5}$, $N_{3,-1}$, $N_{3,-2}$, $N_{3,-5}$,

$N_{4,-2}$, $N_{4,-3}$, $N_{4,-5}$, $N_{4,-6}$, $N_{5,-1}$, $N_{5,-3}$,

$N_{5,-6}$, $N_{6,-4}$, $N_{6,-5}$, $N_{-1,-3}$, $N_{-1,-5}$, $N_{-1,-6}$,

$N_{-2,-6}$, $N_{-3,-5}$.

We saw that, when we have roots from $\mathfrak{r}(\alpha) - k' \mathfrak{r}(\beta)$ to $\mathfrak{r}(\alpha) + k\,\mathfrak{r}(\beta)$,

$$N_{\alpha\beta} = \pm\sqrt{\{\tfrac{1}{2}\,k\,(k'+1)\}}\;|\mathfrak{r}(\beta)|\;.$$

With regard to the ambiguity of sign we note that the relations

$$N_{\alpha\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} = -N_{-\alpha,-\beta} \qquad (4.9)$$

do not link $\mathfrak{r}(\alpha)$ and $\mathfrak{r}(\beta)$ with any $\mathfrak{r}(\gamma)$ in the same half-plane. So we can choose the + and - signs independently for every pair of roots in the half-plane whose sum is a root. For SU_3 we have only one choice because only $\mathfrak{r}(1) + \mathfrak{r}(3)$ is acceptable. Having determined N_{13} the others will come from (4.9) and let us remark that in the subscripts $1 + 3 = 2$. Since neither $\mathfrak{r}(1) - \mathfrak{r}(3)$ nor

$\underline{r}(1) + 2\,\underline{r}(3)$ is a root, $k' = 0$ and $k = 1$ and choosing the positive sign of the square root we have

$$N_{13} = \sqrt{(\tfrac{1}{2})}\,|\underline{r}(3)| = \sqrt{(1/6)}\ .$$

~~The values of the other N's are consequences of (4.9).~~ For the B_2-group we have two choices of sign and for G_2 we have five choices. We readily see that we can arrive at the following results:

$$SU_3:\ \frac{1}{\sqrt{6}} = N_{13} = -N_{1,-2} = -N_{2,-1} = N_{2,-3} = N_{3,-2} = -N_{-1,-3}$$

$$B_2:\ \sqrt{\frac{1}{6}} = N_{13} = -N_{14} = -N_{1,-2} = N_{1,-3} = -N_{2,-1} = -N_{2,-3}$$

$$= N_{3,-1} = N_{3,-2} = N_{3,-4} = -N_{4,-3} = -N_{-1,-3} = N_{-1,-4}$$

$$G_2:\ \frac{1}{2\sqrt{3}} = -N_{13} = N_{16} = N_{1,-2} = -N_{1,-5} = N_{26} = -N_{2,-3} = \tag{4.10}$$

$$= N_{2,-4} = -N_{3,5} = -N_{3,-2} = -N_{-4,-2} = N_{4,-3} =$$

$$= -N_{4,-5} = N_{4,-6} = -N_{5,-1} = N_{5,-6} = N_{6,-4}\ ,\ N_{6,-5} =$$

$$= N_{-1,-3} = -N_{-1,-6} = -N_{-2,-6} = N_{-3,-5}$$

$$\sqrt{\frac{1}{6}} = N_{15} = -N_{1,-3} = -N_{3,-1} = N_{3,-5} = N_{5,-3} = -N_{-1,-5}\ .$$

These values of $N_{\alpha\beta}$ and values of $(r_1(\alpha), r_2(\alpha))$ shown in figures 6, 7, 8 completely specify the commutation relations

$$[H_i, E_\alpha] = r_i(\alpha)\,E_\alpha\ , \qquad [E_\alpha, E_{-\alpha}] = r_i(\alpha)\,H_i\ , \qquad (i = 1, 2)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\,E_{\alpha+\beta}\ .$$

We notice that the order of a Lie group is the rank 2 plus the number of roots: for example, in the SU_3 group the Lie operators are H_1, H_2, E_1, E_2, E_3, E_{-1}, E_{-2}, E_{-3} so that the order is 8. Similarly the order of B_2 is 10 and the order of G_2 is 14. To express this differently: the regular representation of SU_3 is eight-dimensional, that of B_2 is ten-dimensional, and that of G_2 is fourteen-dimensional.

CHAPTER V:

Weights

In discussing Lie algebras it was mentioned that one often represents the operators L_A by square matrices. In ℓ-dimensional representation H_1 and H_2 will be matrices of ℓ rows and ℓ columns. Since H_1 and H_2 commute, it will be possible to find a vector ψ with ℓ components which is an eigenvector of both H_1 and H_2. Hence

$$H_1 \psi = m_1 \psi , \qquad H_2 \psi = m_2 \psi$$

and we may regard (m_1, m_2) as a vector $\mathbf{m}$ in two-dimensional space - the same space as that of $\mathbf{r}(\alpha)$, $\mathbf{r}(\beta)$, ... We call $\mathbf{m}$ the weight of ψ. A weight is <u>simple</u>, if it belongs to only one eigenvector.

It may be seen immediately that the weight of the vector $H_\ell \psi$ is $\mathbf{m}$, because

$$H_1 (H_\ell \psi) = H_\ell H_1 \psi = m_1 H \psi$$

$$H_2 (H_\ell \psi) = m_2 H_\ell \psi .$$

Moreover, since

$$[H_1 , E_\alpha] = r_1(\alpha) E_\alpha ,$$

we have

$$H_1 E_\alpha \psi = E_\alpha H_1 \psi + r_1(\alpha) E_\alpha \psi = (m_1 + r_1(\alpha)) E_\alpha \psi$$

$$H_2 E_\alpha \psi = (m_2 + r_2(\alpha)) E_\alpha \psi ,$$

so $E_\alpha \psi$ is an eigenvector with weight $\underline{m} + \underline{r}(\alpha)$, unless $E_\alpha \psi$ vanishes. Similarly $E_{-\alpha} \psi$ is an eigenvector with weight $\underline{m} - \underline{r}(\alpha)$, unless $E_{-\alpha} \psi$ vanishes. The E_α's are displacement operators in the sense that, when they act on the ℓ-dimensional eigenvectors of the H_i's , they produce other eigenvectors with new weights. They are a generalization to two dimensions of the raising and lowering operators τ_+ , τ_- of SU_2 .

In a representation where both H_1 and H_2 are diagonal their common eigenvectors may be written as

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \ldots .$$

If E_α carries v_1 into an eigenvector with the same weight as v_2 and perhaps makes other changes, we may represent it by the matrix as shown with k_1 real and positive. Since $E_{-\alpha}$ carries v_2 into

$$E_\alpha = k_1 \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad E_{-\alpha} = k_2 \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

an eigenvector with the same weight as v_1, we can represent it as shown with k_2 real and positive. We can take the constant multiplier in each case to be $\sqrt{(k_1 k_2)}$. Each is then the transpose of the other and, since the elements are real,

$$E_{-\alpha} = E_\alpha^{+} . \qquad (5.1)$$

We have seen that by operating with E_α we raise the weight of the eigenvector by $\underset{\sim}{r}(\alpha)$. The operators H_1 and H_2 being represented by ℓ-dimensional matrices have ℓ eigenvalues, so the number of possible weights is finite. Thus a state $\psi(M)$ with weight $\underset{\sim}{M}$ will be reached such that $E_\alpha \psi(\underset{\sim}{M})$ vanishes. We suppose $\psi(\underset{\sim}{M})$ to be normalized and we may put

$$E_{-\alpha} \psi(\underset{\sim}{M}) = a \, \psi(\underset{\sim}{M} - \underset{\sim}{r}(\alpha)) ,$$

where a is a real constant introduced in order to have $\psi(\underset{\sim}{M} - \underset{\sim}{r}(\alpha))$ normalized. Since

$$[E_\alpha , E_{-\alpha}] = r_i(\alpha) H_i = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{H}) ,$$

and since

$$(\psi^{+}(\underset{\sim}{M}) [E_\alpha , E_{-\alpha}] \psi(\underset{\sim}{M})) = (\psi^{+}(\underset{\sim}{M}) E_\alpha E_{-\alpha} \psi(\underset{\sim}{M})) =$$

$$= (\psi^{+}(\underset{\sim}{M}) E_{-\alpha}^{+} E_{-\alpha} \psi(\underset{\sim}{M})) = a^2 (\psi^{+}(\underset{\sim}{M} - \underset{\sim}{r}(\alpha)) \psi(\underset{\sim}{M} - \underset{\sim}{r}(\alpha)) = a^2 ,$$

$$(\psi^{+}(\underset{\sim}{M}) (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{H}) \psi(\underset{\sim}{M})) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{M}) \psi^{+}(M) \psi(M) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{M}) ,$$

we deduce that

$$a^2 = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{M}) . \qquad (5.2)$$

We investigate how the weight vectors $\underset{\sim}{m}$ may be systematically related to the root vectors $\underset{\sim}{r}(\alpha)$. We write

$$H_1\,\psi(\underset{\sim}{m}) = m_1\,\psi(\underset{\sim}{m}), \quad H_2\,\psi(\underset{\sim}{m}) = m_2\,\psi(\underset{\sim}{m})$$

$$E_\alpha\,\psi(\underset{\sim}{m}) = c_{\underset{\sim}{m}}\,\psi(\underset{\sim}{m} + \underset{\sim}{r}(\alpha)), \quad E_{-\alpha}\,\psi(\underset{\sim}{m}) = d_{\underset{\sim}{m}}\,\psi(\underset{\sim}{m} - \underset{\sim}{r}(\alpha)) .$$

By raising and lowering the wieghts we arrive at integers j and j', positive or zero, such that

$$E_\alpha\,\psi(\underset{\sim}{m} + j\,\underset{\sim}{r}(\alpha)) = 0, \quad E_{-\alpha}\,\psi(\underset{\sim}{m} - j'\,\underset{\sim}{r}(\alpha)) = 0,$$

which yield

$$c_{\underset{\sim}{m}+j\underset{\sim}{r}(\alpha)} = 0, \quad d_{\underset{\sim}{m}-j'\underset{\sim}{r}(\alpha)} = 0. \tag{5.3}$$

The relation

$$(E_\alpha E_{-\alpha} - E_{-\alpha} E_\alpha)\,\psi(\underset{\sim}{m}) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{H})\,\psi(\underset{\sim}{m}) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m})\,\psi(\underset{\sim}{m})$$

gives

$$E_\alpha\, d_m\,\psi(\underset{\sim}{m} - \underset{\sim}{r}(\alpha)) - E_{-\alpha}\, c_{\underset{\sim}{m}}\,\psi(\underset{\sim}{m} + \underset{\sim}{r}(\alpha)) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m})\,\psi(\underset{\sim}{m})$$

$$d_{\underset{\sim}{m}}\, c_{\underset{\sim}{m}-\underset{\sim}{r}(\alpha)}\,\psi(\underset{\sim}{m}) - d_{\underset{\sim}{m}+\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}}\,\psi(\underset{\sim}{m}) = (\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m})\,\psi(\underset{\sim}{m}),$$

so

$$d_{\underset{\sim}{m}+\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}} - d_{\underset{\sim}{m}}\, c_{\underset{\sim}{m}-\underset{\sim}{r}(\alpha)} = -\,(\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m}) .$$

We replace $\underset{\sim}{m}$ successively by $\underset{\sim}{m} + j\,\underset{\sim}{r}(\alpha)$, $\underset{\sim}{m} + (j-1)\,\underset{\sim}{r}(\alpha)$, ... $\underset{\sim}{m} - j'\,\underset{\sim}{r}(\alpha)$ and using (5.3) obtain

$$\begin{array}{ccccl}
0 & - & d_{\underset{\sim}{m}+j\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}+(j-1)\underset{\sim}{r}(\alpha)} & = & -\,(\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m}) - j\,|\underset{\sim}{r}(\alpha)|^2 \\
d_{\underset{\sim}{m}+j\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}+(j-1)\underset{\sim}{r}(\alpha)} & - & d_{\underset{\sim}{m}+(j-1)\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}+(j-2)\underset{\sim}{r}(\alpha)} & = & -\,(\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m}) - (j-1)\,|\underset{\sim}{r}(\alpha)|^2 \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
d_{\underset{\sim}{m}-(j'-1)\underset{\sim}{r}(\alpha)}\, c_{\underset{\sim}{m}-j'\underset{\sim}{r}(\alpha)} & - & 0 & = & -\,(\underset{\sim}{r}(\alpha) \cdot \underset{\sim}{m}) + j'\,|\underset{\sim}{r}(\alpha)|^2 ,
\end{array}$$

so that on summing

$$0 = -(j+j'+1)\,(\mathbf{r}(\alpha)\cdot\mathbf{m}) + \tfrac{1}{2}(j+j'+1)\,(j'-j)\,\|\mathbf{r}(\alpha)\|^2$$

$$\frac{2\,(\mathbf{r}(\alpha)\cdot\mathbf{m})}{\|\mathbf{r}(\alpha)\|^2} = j' - j\,, \qquad (5.4)$$

an integer. Moreover

$$\mathbf{m} - \frac{2\,(\mathbf{r}(\alpha)\cdot\mathbf{m})}{\|\mathbf{r}(\alpha)\|^2}\,\mathbf{r}(\alpha) = \mathbf{m} - (j'-j)\,\mathbf{r}(\alpha)\,, \qquad (5.5)$$

which is a weight since it lies on the sequence between $\mathbf{m} - j'\,\mathbf{r}(\alpha)$ and $\mathbf{m} + j\,\mathbf{r}(\alpha)$. The weights may be found by calculating the possible values of $\mathbf{m}$ that satisfy (5.4) for the different $\mathbf{r}(\alpha)$'s in the root diagram. Equation (5.5) shows that we may go from one weight to another by reflecting $\mathbf{m}$ in a line perpendicular to any root $\mathbf{r}(\alpha)$, just as was illustrated in Figure 5 for the reflection of root vectors. Such reflections are called <u>Weyl reflections</u> and the weights so related are called <u>equivalent weights</u>.

CHAPTER VI:

Weight Diagrams

We apply the foregoing theorems to the construction of <u>weight diagrams</u>, that is, diagrams on which points with coordinates (m_1, m_2) are plotted. We examine in turn the SU_3, B_2, G_2 groups.

SU_3 Group.

Figure 6 shows that the roots are

$$\underset{\sim}{r}(1) = (\frac{1}{\sqrt{3}}, 0), \quad \underset{\sim}{r}(2) = (\frac{1}{2\sqrt{3}}, \frac{1}{2}), \quad \underset{\sim}{r}(3) = (-\frac{1}{2\sqrt{3}}, \frac{1}{2}),$$

so

$$\frac{2\underset{\sim}{r}(1)}{\|\underset{\sim}{r}(1)\|^2} = (2\sqrt{3}, 0), \quad \frac{2\underset{\sim}{r}(2)}{\|\underset{\sim}{r}(2)\|^2} = (\sqrt{3}, 3), \quad \frac{2\underset{\sim}{r}(3)}{\|\underset{\sim}{r}(3)\|^2} = (-\sqrt{3}, 3).$$

Thus the theorem that $\frac{2(\underset{\sim}{m}.\underset{\sim}{r}(\alpha))}{\|\underset{\sim}{r}(\alpha)\|^2}$ is an integer for $\alpha = 1, 2, 3$

gives $2\sqrt{3}\, m_1 =$ integer, $\sqrt{3}\, m_1 + 3m_2 =$ integer, $-\sqrt{3}\, m_1 + 3m_2 =$ integer.

Since $2\sqrt{3}\, m_1 \pm 6m_2$ are even integers,

we may write

$2\sqrt{3}\, m_1 = \lambda + \mu$, $\quad 6m_2 = \lambda - \mu \quad (\lambda, \mu$ integers$)$

$$\underset{\sim}{m} = \lambda(\frac{\sqrt{3}}{6}, \frac{1}{6}) + \mu(\frac{\sqrt{3}}{6}, -\frac{1}{6}). \qquad (6.1)$$

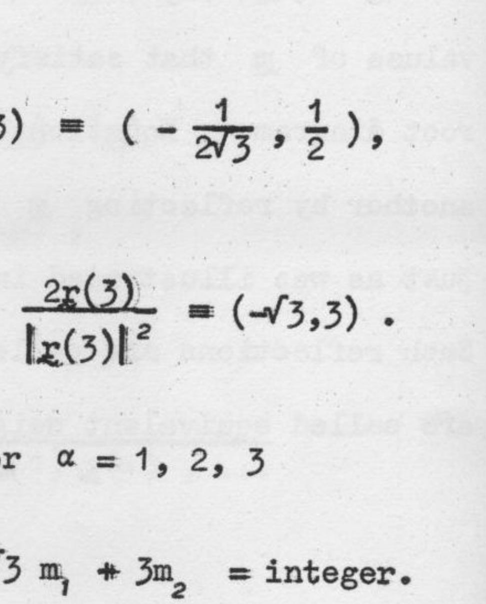


Fig. 9. The lines perpendicular to the root vectors of SU_3.

For the purpose of obtaining equivalent weights we draw the lines perpendicular to the root vectors. We then take different sets of values of the integers λ and μ in (6.1).

i) $\lambda = 0$, $\mu = 0$.

The weight $\underline{m}$ is zero and reflections give nothing else. There is only one weight and we say that we have the one-dimensional representation $D^{(1)}(0, 0)$.

ii) $\lambda = 1$, $\mu = 0$.

One weight is $(\frac{\sqrt{3}}{6}, \frac{1}{6})$ and on reflection we obtain $(-\frac{\sqrt{3}}{6}, \frac{1}{6})$ and $(0, -\frac{1}{3})$. We see that we can go from any weight to another by adding an $\underline{r}(\alpha)$. We have three wave functions, eigenfunctions of H_1 and H_2 , related to the three weights. The matrices representing the displacement operators will therefore be three-dimensional, as will be those representing H_1 and H_2 . Hence we have a three-dimensional representation of the Lie operators which we denote by $D^{(3)}(1, 0)$.

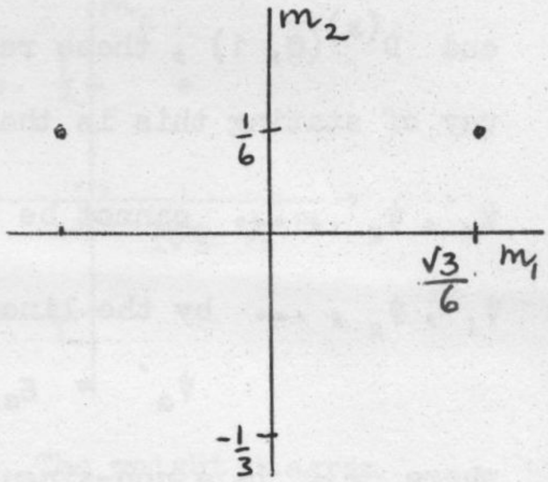


Fig. 10. The weight diagram for $D^{(3)}(1,0)$ of SU_3 .

iii) $\lambda = 0$, $\mu = 1$.

This is the three-dimensional representation $D^{(3)}(0, 1)$ whose weight diagram is obtained from that of $D^{(3)}(1, 0)$ by reflection in the m_1-axis. The weights of $D^{(3)}(0, 1)$ have minus the values for $D^{(3)}(1, 0)$.

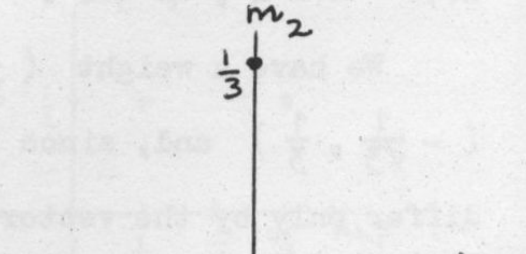


Fig. 11. The weight diagram of $D^{(3)}(0,1)$ of SU_3 .

Two representations are said to be equivalent, if the matrices representing the operators are related by a similarity transformation. Such a transformation leaves eigenvalues unaltered, because if

$$\ell \psi = a \psi$$

and

$$\ell' = S^{-1} \ell S \ , \qquad S^{-1} S = S S^{-1} = 1 \ ,$$

it follows that

$$\ell' (S^{-1} \psi) = S^{-1} \ell S S^{-1} \psi = S^{-1} a \psi = a (S^{-1} \psi) \ .$$

Since the eigenvalues of H_1 and H_2 are different for $D^{(3)}(1, 0)$ and $D^{(3)}(0, 1)$, these representations are inequivalent. Another way of stating this is that eigenfunctions in one representation

$\psi_1', \psi_2', \ldots$ cannot be expressed in terms of those in the other

$\psi_1, \psi_2, \ldots$ by the linear relation

$$\psi_a' = g_{ab} \psi_b \ ,$$

where g_{ab} is a non-singular matrix. Otherwise by taking

$$S^{-1}_{ab} = g_{ab}$$

we are back to the case of equivalent representations.

iv) $\lambda = 2$, $\mu = 0$.

We have a weight $\left(\frac{1}{\sqrt{3}}, \frac{1}{3} \right)$, by reflection $\left(-\frac{1}{\sqrt{3}}, \frac{1}{3} \right)$ and, since the adjacent weights can differ only by the vector $\underline{r}(1)$ i.e. $\left(\frac{1}{\sqrt{3}}, 0 \right)$, there must be a weight $\left(0, \frac{1}{3}\right)$. From this we get $\left(\pm \frac{1}{2\sqrt{3}}, -\frac{1}{6} \right)$ by reflection and from $\left(\frac{1}{\sqrt{3}}, \frac{1}{3} \right)$ we get $\left(0, -\frac{2}{3}\right)$ by reflection.

We have the six-dimensional representation $D^{(6)}(2,0)$.

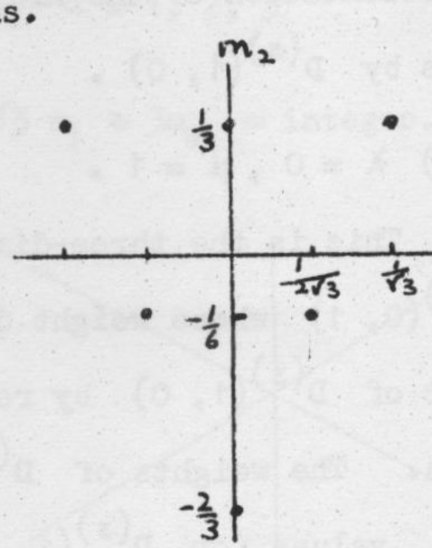


Fig. 12. The weight diagram for $D^{(6)}(2,0)$ of SU_3.

v) $\lambda = 0\,,\ \mu = 2\,.$

This is $D^{(6)}(0,2)$. The weight diagram is the reflection in the m_1-axis of the previous one, and the two representations are inequivalent.

vi) $\lambda = 1\,,\ \mu = 1\,.$

We have a weight $\left(\frac{1}{\sqrt{3}}\,,\,0\right)$ and on reflecting we obtain $\left(\pm\frac{1}{2\sqrt{3}}\,,\,\pm\frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{3}}\,,\,0\right)$. There is also a weight at the origin. Actually we shall later obtain two independent wave functions in this representation with zero weight; zero weight is not simple. Thus we have $D^{(8)}(1,1)$. It is the only eight-dimensional representation, so it is the regular representation of the group.

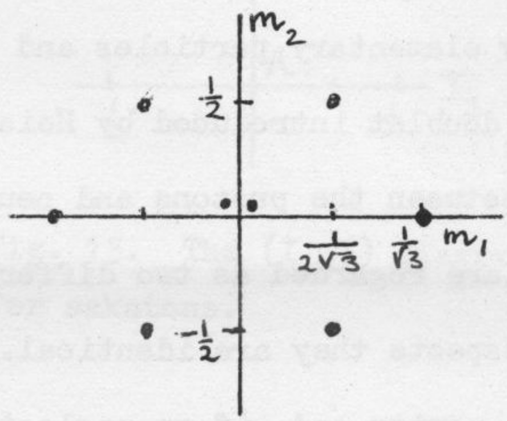


Fig. 13. The weight diagram for $D^{(8)}(1,1)$ of SU_3.

vii) $\lambda = 3\,,\ \mu = 0\,.$

There is a weight $\left(\frac{\sqrt{3}}{2}\,,\,\frac{1}{2}\right)$ and proceeding as before we obtain a total of ten weights as shown in Figure 14, so the representation is $D^{(10)}(3,\,0)$.

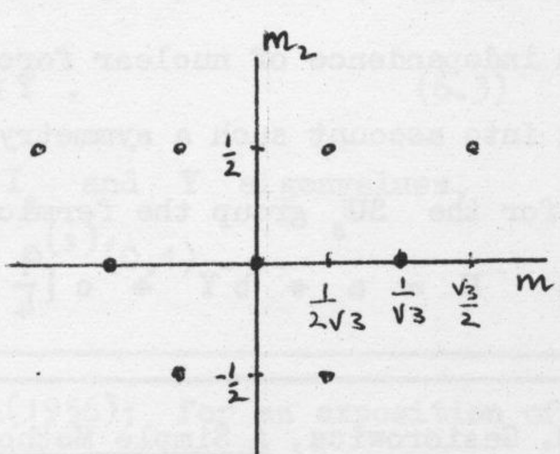


Fig. 14. The weight diagram for $D^{(10)}(3,0)$ of SU_3.

viii) $\lambda = 0$, $\mu = 3$.

This is the representation $D^{(10)}(0, 3)$ inequivalent to the preceding one. The weight diagram is the reflection in the m_1-axis of Figure 14.

An exhaustive study of the properties of the weight diagrams for the SU_3 group has been made by Gasiorowicz[1].

Before finding weight diagrams for the other Lie groups of rank two we shall indicate how some of the above weight diagrams may be used to classify elementary particles and resonances. We recall the notion of isospin doublet introduced by Heisenberg to describe charge exchange forces between the protons and neutrons in a nucleus. The proton and neutron are regarded as two different isospin states of the nucleon; in other respects they are identical. They have the same mechanical spin, the same parity and, if we neglect the weaker electromagnetic forces, the same mass. Similarly when we place particles in a supermultiplet we imply that they all have the same spin and parity, and that they would have the same mass but for some symmetry-breaking interaction that upsets the invariance property of the group in a way analogous to the violation of the charge independence of nuclear forces by the electromagnetic interaction. Taking into account such a symmetry-breaking interaction Okubo[2] has given for the SU_3 group the fermion mass formula

$$M = a + bY + c\left\{\frac{1}{4}Y^2 - I(I+1)\right\}, \qquad (6.2)$$

1. S. Gasiorowicz, <u>A Simple Method in the Analysis of SU_3</u> (Argonne National Laboratory - 6729, June 1963).

2. S. Okubo, <u>Prog. Theor. Phys.</u> <u>27</u>, 949 (1962).

where M is the mass, Y the hypercharge, I the isospin and a, b, c are independent of Y and I but depend on the representation.

The $D^{(3)}(1, 0)$ weight diagram suggests the Sakata model[3], in which the fundamental constituents are p, n, Λ and their antiparticles $\bar{p}$, $\bar{n}$, $\bar{\Lambda}$. The other elementary particles are expressed as bound states of these as follows:

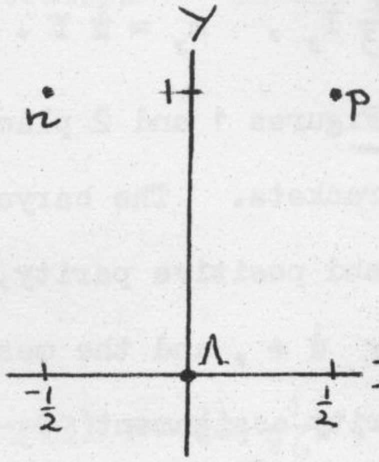


Fig. 15. The (I_3, Y) diagram for sakatons.

$$\pi^+ = p\bar{n}, \quad \pi^- = \bar{p}n, \quad \pi^0 = \frac{1}{\sqrt{2}}(p\bar{p}-n\bar{n}),$$

$$\eta = \frac{1}{\sqrt{2}}(p\bar{p}+n\bar{n}),$$

$$K^+ = p\bar{\Lambda}, \quad K^0 = n\bar{\Lambda}, \quad K^- = \bar{p}\Lambda, \quad \bar{K^0} = \bar{n}\Lambda,$$

$$\Sigma^+ = p\bar{n}\Lambda, \quad \Sigma^- = \bar{p}n\Lambda, \quad \Sigma^0 = \frac{1}{\sqrt{2}}(p\bar{p}-n\bar{n})\Lambda,$$

$$\Xi^0 = \bar{n}\Lambda\Lambda, \quad \Xi^- = \bar{p}\Lambda\Lambda.$$

The sakatons p, n, Λ have the same spin $\frac{1}{2}$ and positive parity. We may identify their (I_3, Y) values with the weights of $D^{(3)}(1, 0)$ by writing

$$m_1 = \frac{1}{\sqrt{3}} I_3, \qquad m_2 = -\frac{1}{3} + \frac{1}{2}Y. \tag{6.3}$$

Since the antiparticles have the opposite I_3 and Y eigenvalues, we may place them on the weight diagram of $D^{(3)}(0,1)$.

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3. S. Sakata, Prog. Theor. Phys. 16, 686 (1956); for an exposition of later developments vide Suppl. Prog. Theor. Phys. 19 (1961).

The $D^{(8)}(1, 1)$ weight diagram accommodates the baryon octet and the meson octet, if we put

$$m_1 = \frac{1}{\sqrt{3}} I_3 \;, \quad m_2 = \tfrac{1}{2} Y \;. \qquad (6.4)$$

We combine Figures 1 and 2 placing the mesons in brackets. The baryons have all spin $\frac{1}{2}$ and positive parity, which we denote by $\frac{1}{2}+$, and the mesons have spin and parity assignment $0-$. There is also an incomplete baryon resonance $\frac{5}{2}+$ octet and a well-established vector meson resonance $1-$ nonet as shown in Figure 17 with the relations (6.4) obeyed. It is thought that the nonet is a combination of the SU_3 singlet and octet, the ω^0 with mass 783 MeV and the ϕ^0 with mass 1020 MeV belonging to both representations.

In the $D^{(10)}(3,0)$ weight diagram we substitute from (6.4) and find a $\frac{3}{2}+$ decuplet picture as shown in Figure 18. A set of nine resonances, the Δ, Y^*, Ξ^* with charge and mass as indicated was known, so one suspected the existence of an isospin singlet with hypercharge -2 and therefore strangeness -3. Moreover the coefficients a, b, c in (6.2)

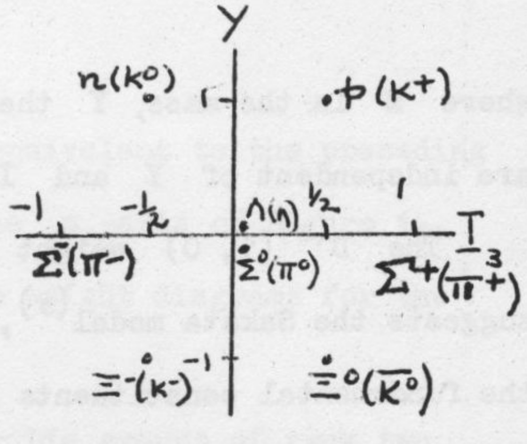

Fig. 16. The baryon (meson) octet.

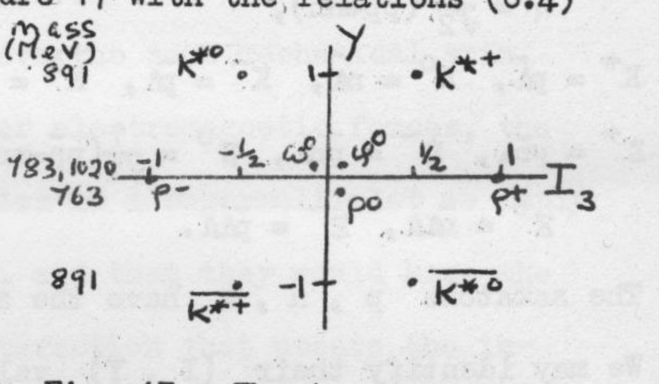

Fig. 17. The 1-meson resonance nonet.

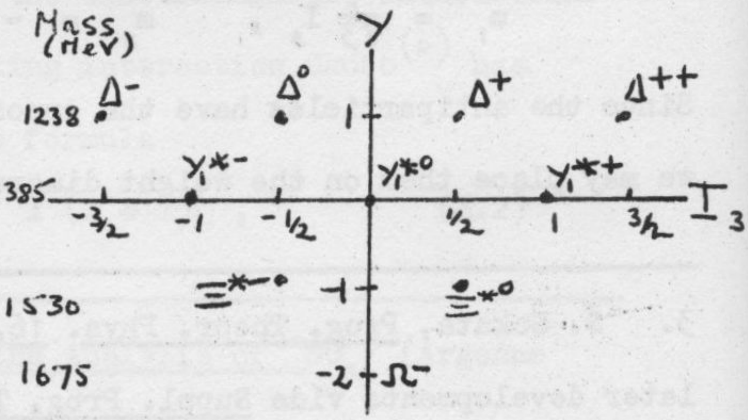

Fig. 18. The $\frac{3}{2}+$ decuplet.

could be determined from the masses of the resonances, and one could deduce the mass of the missing particle. A particle answering to its description called Ω^- was found by Barnes and his collaborators [4] in the process

$$K^- + p \to \Omega^- + K^+ + K^o .$$

B_2 Group.

The roots shown in Figure 7 are

$$\underset{\sim}{r}(1) = (\tfrac{1}{\sqrt{6}}, 0), \quad \underset{\sim}{r}(2) = (\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}), \quad \underset{\sim}{r}(3) = (0, \tfrac{1}{\sqrt{6}}),$$

$$\underset{\sim}{r}(4) = (-\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}})$$

and the theorem that $\dfrac{2\,(\underset{\sim}{m} \cdot \underset{\sim}{r}(\alpha))}{\|\underset{\sim}{r}(\alpha)\|^2}$ is an integer gives

$$2\sqrt{6}\, m_1 = \text{integer}, \qquad \sqrt{6}\, m_1 + \sqrt{6}\, m_2 = \text{integer},$$

$$2\sqrt{6}\, m_2 = \text{integer}, \qquad -\sqrt{6} m_1 + \sqrt{6}\, m_2 = \text{integer}.$$

These are satisfied, if

$$2\sqrt{6}\, m_1 = c + d, \qquad 2\sqrt{6}\, m_2 = c - d$$

with c, d integers. Hence

$$2\sqrt{6}\, \underset{\sim}{m} = (c - d)(1, 1) + 2d(1, 0)$$

or

$$\underset{\sim}{m} = \frac{\lambda}{2\sqrt{6}}(1, 1) + \frac{\mu}{\sqrt{6}}(1, 0),$$

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4. V. E. Barnes et al. Phys. Rev. Lett. 12, 204 (1964).

where λ and μ are integers. The lines through which we reflect are just the root vectors themselves.

i) $\lambda = 0$, $\mu = 0$.

This is the one-dimensional representation $D^{(1)}(0,0)$ with zero weight.

ii) $\lambda = 1$, $\mu = 0$.

One weight is $(\frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{6}})$ and by reflection we obtain altogether

$(\pm\frac{1}{2\sqrt{6}}, \pm\frac{1}{2\sqrt{6}})$, so we have $D^{(4)}(1,0)$.

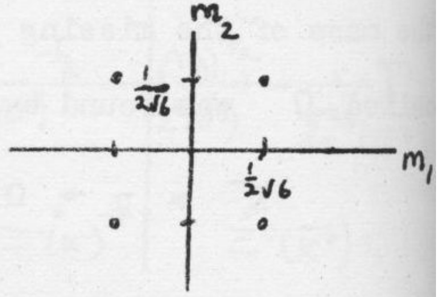

Fig. 19. The weight diagram for $D^{(4)}(1,0)$ of B_2.

iii) $\lambda = 0$, $\mu = 1$.

One weight is $(\frac{1}{\sqrt{6}}, 0)$ and on reflecting we get $(0, \frac{1}{\sqrt{6}})$, $(-\frac{1}{\sqrt{6}}, 0)$ $(0, -\frac{1}{\sqrt{6}})$. If we displace $(-\frac{1}{\sqrt{6}}, 0)$ by $\underset{\sim}{r}(1)$ we arrive at the origin, so we have a five-dimensional representation $D^{(5)}(0,1)$.

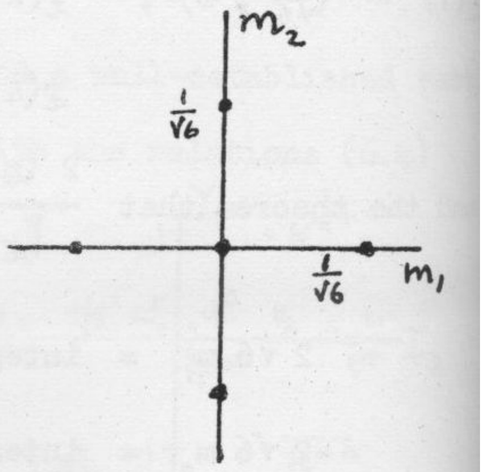

Fig. 20. The weight diagram for $D^{(5)}(0,1)$ of B_2.

iv) $\lambda = 2$, $\mu = 0$.

One weight is $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ and by reflection we obtain $(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$, $(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$, $(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$. Since the weights can differ from each other only by a root vector, we must have in addition weights at $(0, \frac{1}{\sqrt{6}})$, $(-\frac{1}{\sqrt{6}}, 0)$, $(0, -\frac{1}{\sqrt{6}})$, $(\frac{1}{\sqrt{6}}, 0)$, $(0, 0)$. The weight at the

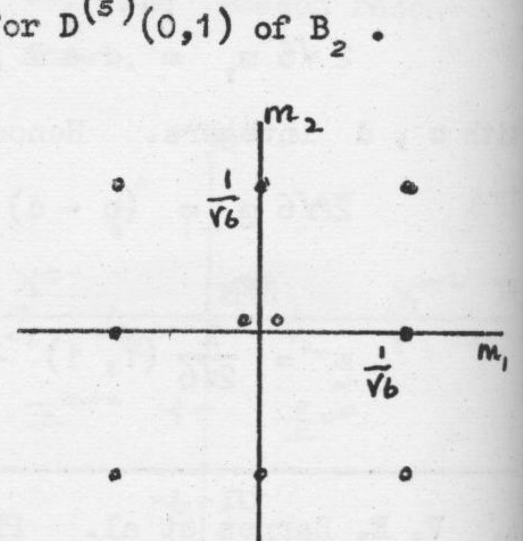

Fig. 21. The weight diagram for $D^{(10)}(2,0)$ of B_2.

origin is double, since we shall later obtain two independent wave functions with zero weight. We thus have a ten-dimensional representation $D^{(10)}(2, 0)$. It is the regular representation of the group.

The $D^{(4)}(1, 0)$ weight diagram may be employed to depict the leptons. These particles have spin $\frac{1}{2}$ and their parity does not interest us, since interactions in which leptons alone take part are weak and therefore parity is not conserved. We put muon displacement $= \sqrt{6}\, m_1$, charge displacement $= \sqrt{6}\, m_2$, where muon and charge displacement have been defined in Chapter I, and obtain Figure 3 for the leptons.

A group intimately related to B_2 is the symplectic group C_2. The root and weight diagrams are the same as those of B_2 but turned through an angle $\pi/4$.

G_2 Group.

The weight in this case is

$m = \lambda\left(\frac{1}{2\sqrt{3}}, 0\right) + \mu\left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)$. We draw the weight diagrams for $D^{(7)}(1,0)$ and $D^{(14)}(0,1)$. The latter is the regular representation.

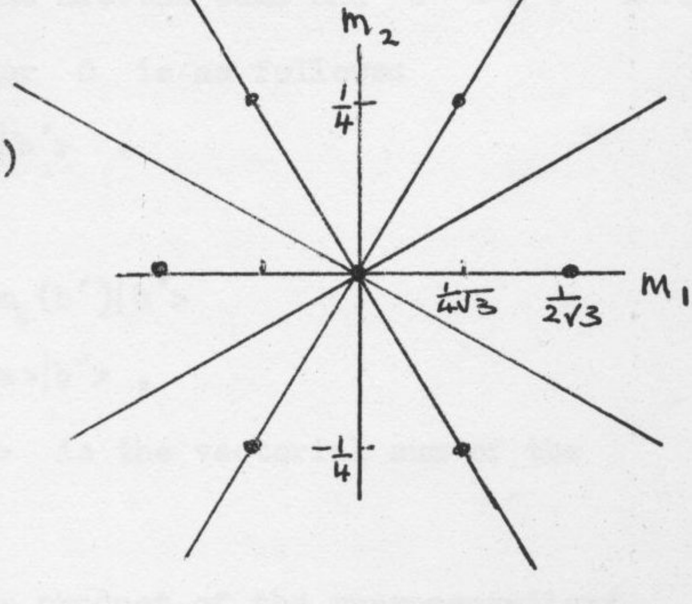

Fig. 22. The weight diagram for $D^{(7)}(1,0)$ of G_2.

The seven-dimensional representation can describe the seven baryons $(p, n, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-)$ with

$$m_1 = \frac{1}{2\sqrt{3}} I_3 , \quad m_2 = \frac{1}{4} Y .$$

The Λ particle is assigned to the one-dimensional representation $D^{(1)}(0, 0)$. The seven mesons $(K^+, K^0, \pi^+, \pi^0, \pi^-, \overline{K^0}, K^-)$ are likewise referred to $D^{(7)}(1,0)$. The scheme would allow opposite parities for Σ and Λ.

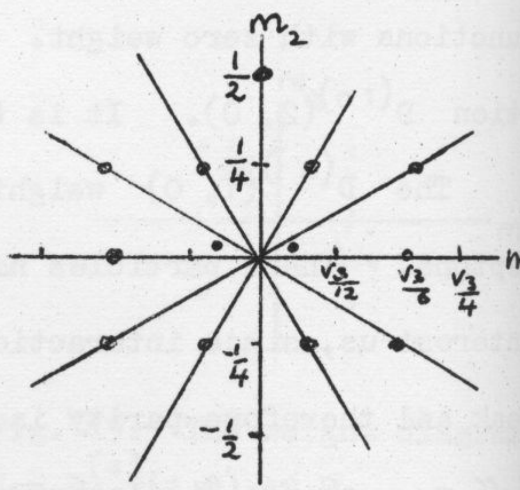

Fig. 23. The weight diagram for $D^{(14)}(0,1)$ of G_2.

CHAPTER VII:

Product Representations

When dealing with the SU_2 group we saw that the decomposition of the product of two doublet spin states

$$D^{(2)} \otimes D^{(2)} = D^{(1)} \oplus D^{(3)}$$

may be represented geometrically as in Figure 4. We shall explain now how weight diagrams may be used to give the decomposition of the product of two representations, at least in some simpler but important cases.

Let $|a\rangle$, $|b'\rangle$ be normalised ket vectors for two representations of the same Lie group. The two representations may be the same or different. The product $|a\rangle|b'\rangle$ is treated mathematically like the product of the spin wave functions of two electrons. Thus $|b'\rangle|a\rangle$ is not the same as $|a\rangle|b'\rangle$ unless the two representations are the same and $b' = a$. Moreover the effect of applying an operator O is as follows:

$$O|a\rangle|b'\rangle = O|a\rangle . |b'\rangle + |a\rangle . O|b'\rangle \quad .$$

In particular

$$\begin{aligned} H_i|a\rangle|b'\rangle &= m_i(a)|a\rangle|b'\rangle + |a\rangle m_i(b')|b'\rangle \\ &= (m_i(a) + m_i(b'))\ |a\rangle|b'\rangle , \end{aligned}$$

that is to say, the weight of $|a\rangle|b'\rangle$ is the vectorial sum of the weights of $|a\rangle$ and $|b'\rangle$.

As an illustration we examine the product of the representations $D^{(3)}(1, 0)$ and $D^{(3)}(0, 1)$ of SU_3 . The weight diagrams are shown in Figures 10 and 11. If we displace the weight diagram of $D^{(3)}(1, 0)$ by the weight $(0, \frac{1}{3})$ we obtain the weights

$(\frac{\sqrt{3}}{6}, \frac{1}{2})$, $(-\frac{\sqrt{3}}{6}, \frac{1}{2})$, $(0, 0)$.

On displacing successively by the three weights of $D^{(3)}(0, 1)$ we obtain the nine weights shown in Figure 24. On referring to Figure 13 we see that we have all the weights of $D^{(8)}(1, 1)$ plus one more at the origin. This is just the weight of $D^{(1)}(0, 0)$, so we have the decomposition

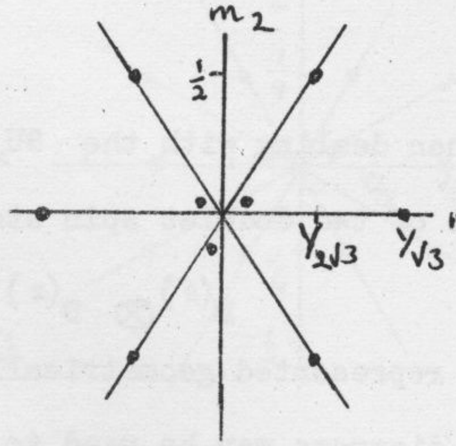

Fig. 24. The weight diagram for the product $D^{(3)}(1,0) \otimes D^{(3)}(0,1)$ of SU_3.

$$D^{(3)}(1, 0) \otimes D^{(3)}(0, 1) = D^{(1)}(0, 0) \oplus D^{(8)}(1, 1). \quad (7.1)$$

The weight diagram for $D^{(8)}(1, 1)$ cannot itself be decomposed into diagrams of lower dimensionality. We say that $D^{(8)}(1, 1)$ is an <u>irreducible representation</u> of SU_3 and (7.1) gives the decomposition of the product of $D^{(3)}(1, 0)$ and $D^{(3)}(0, 1)$ into its irreducible representations. The method can readily be extended to the product of three representations and one may, for example, obtain the decomposition

$$D^{(3)}(1,0) \otimes D^{(3)}(1,0) \otimes D^{(3)}(1,0) =$$
$$= D^{(1)}(0,0) \oplus D^{(8)}(1,1) \oplus D^{(8)}(1,1) \oplus D^{(10)}(3,0). \quad (7.2)$$

As a second illustration we consider the product of the B_2 representation $D^{(4)}(1, 0)$ by itself. Displacing the weight diagram of Figure 19 with respect to its own four weights we obtain the sixteen weights shown in Figure 25. Ten of these are accounted for in Figure 21,

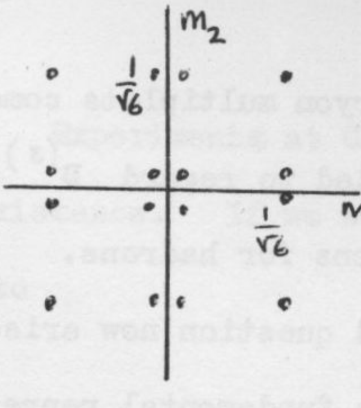

Fig. 25. The weight diagram for the product $D^{(4)}(1,0) \otimes D^{(4)}$ of B_2.

five more in Figure 20 and the remaining zero weight belongs to $D^{(1)}(0, 0)$. In this way we have decomposed the product into the sum of irreducible representations as follows:

$$D^{(4)}(1,0) \otimes D^{(4)}(1,0) = D^{(1)}(0,0) \oplus D^{(5)}(0,1) \oplus D^{(10)}(2,0) . \quad (7.3)$$

The simplest product representation in the G_2 group is $D^{(7)}(1, 0) \otimes D^{(7)}(1, 0)$ and, since this contains forty-nine weights, the geometrical method is not very helpful. Tables showing the irreducible representation arising from product representations for the Lie groups of rank 2 are to be found in the review article of Behrends, Dreitlein, Fronsdal and Lee[1].

The decomposition of product representations of SU_3 raises the question of what is the fundamental representation for strongly interacting particles, on which all the representations classifying the particles are based. We have seen that the boson meson multiplets are the singlet and the octet, and that the fermion baryon multiplets are the octet and the decuplet. In addition there is a singlet $\frac{1}{2}$ - Λ-resonance of mass 1405 MeV denoted by Y_0^*. According to (7.1) the boson multiplets come from the product $D^{(3)}(1, 0) \otimes D^{(3)}(0,1)$, and according to

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1. R. E. Behrends, J. Dreitlein, C. Fronsdal and W. Lee, Rev. Mod.Phys. 34, 1 (1962).

(7.2) the baryon multiplets come from the product of three $D^{(3)}(1, 0)$. Thus we are led to regard $D^{(3)}(1, 0)$ and $D^{(3)}(0, 1)$ as the fundamental representations for hadrons.

A second question now arises of how to interpret physically the particles in the fundamental representations. We have seen that for all meson and baryon supermultiplets we must express (I_3, Y) in terms of (m_1, m_2) by

$$I_3 = \sqrt{3}\, m_1 , \qquad Y = 2 m_2 .$$

By referring to (6.3) we see that the second relation is not obeyed by sakatons, so that the particles are not just p , n , Λ . Let us denote by q the particles of the $D^{(3)}(1, 0)$ representation, so that the antiparticles $\bar{q}$ having opposite values of I_3 and Y are related to $D^{(3)}(0,1)$. Clearly these must have spin $\frac{1}{2}$, since a combination of two of them gives a boson of spin 0 or 1 and a combination of three of them gives a baryon with spin $\frac{1}{2}$ or $\frac{3}{2}$. The mesons are formed from $q\bar{q}$ and the baryons from qqq , so q must have baryon number 1/3. The above relations and

$$Q = I_3 + \tfrac{1}{2} Y \tag{7.4}$$

give for the q 's in the weight diagram

position	I_3	Y	Q
$(\frac{\sqrt{3}}{6}, \frac{1}{6})$	1/2	1/3	2/3
$(-\frac{\sqrt{3}}{6}, \frac{1}{6})$	- 1/2	1/3	- 1/3
$(0, -\frac{1}{3})$	0	- 2/3	- 1/3 .

These hypothetical particles with fractional charge and fractional baryon number are called <u>quarks</u>[2] or <u>aces</u>[3]. Experiments at CERN[4] and at Brookhaven[5] have failed to reveal their existence. If we wish to safeguard integral charge, we can alter (7.4) to

$$Q = I_3 + \frac{1}{2} Y + \frac{1}{3} C ,$$

where C is called the <u>triality</u> or <u>supercharge</u> and has the values 1 or - 2 . It is zero for physical particles.

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2. M. Gell-Mann, <u>Physics Letters</u> <u>8</u>, 214 (1964).

3. G. Zweig, ~~Phys. Rev.~~ CERN Reports 8182/Th 401; 8419/Th 412 (1964).

4. D. R. O. Morrison, <u>Physics Letters</u> <u>9</u>, 199 (1964); H. Bingham, M. Dickenson, R. Diebold, W. Koch, D. W. G. Leith, M. Nikolić, B. Ronne, R. Huson, P. Musset and J. J. Veillet, <u>ibid</u>. <u>9</u>, 201 (1964).

5. L. B. Leipuner, W. T. Chu, R. C. Larsen and R. K. Adair, <u>Phys. Rev. Lett</u>. <u>12</u>, 423 (1964).

CHAPTER VIII:

Explicit Representation of the SU_3 Group

Three-Dimensional Representation

We wish to set up explicit representations of the operators and wave functions that occur in the algebra of the SU_3 group. We start with $D^{(3)}(1,0)$ assigning to the weights

$(\frac{\sqrt{3}}{6}, \frac{1}{6})$, $(-\frac{\sqrt{3}}{6}, \frac{1}{6})$, $(0, -\frac{1}{3})$ the normalized and mutually orthogonal ket vectors

$|\{3\}, 1\rangle$, $|\{3\}, 2\rangle$, $|\{3\}, 3\rangle$,

respectively. These we may also write

ψ_1 , ψ_2 , ψ_3

or

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

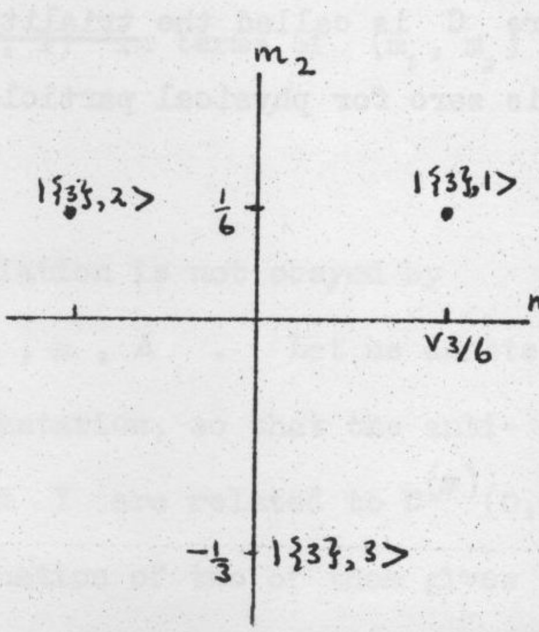


Fig. 26. The ket vectors for the $D^{(3)}(1,0)$ representation of SU_3 .

They are the wave functions of quarks.

Since H_1 , H_2 give the first, second components of the weights

$$H_1 = \frac{\sqrt{3}}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The E_1-operator acting on a ket of weight $\underline{m}$ gives a state of weight $\underline{m} + \underline{r}(1)$, i.e. $\underline{m} + (\frac{\sqrt{3}}{3}, 0)$ if it exists; otherwise it gives zero.

Hence

$$E_1 \,|\{3\}, 1\rangle = 0 \,, \quad E_1 \,|\{3\}, 2\rangle = \text{const.} \,|\{3\}, 1\rangle \,, \quad E_1|\{3\}, 3\rangle = 0 \,,$$

$$E_{-1}|\{3\}, 1\rangle = \text{const.} \,|\{3\}, 2\rangle \,, \quad E_{-1}|\{3\}, 2\rangle = 0 \,, \quad E_{-1}|\{3\}, 3\rangle = 0 \,.$$

Let us write

$$E_{-1} \,|\{3\}, 1\rangle = c \,|\{3\}, 2\rangle$$

and, since $E_1 \,|\{3\}, 1\rangle$ vanishes, equation (5.2) gives

$$c^2 = \frac{\sqrt{3}}{3}\frac{\sqrt{3}}{6} = \frac{1}{6} \,.$$

Since by (5.1) $E_1 = E_{-1}^{+}$, we write

$$E_{-1} = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \qquad E_1 = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and similarly

$$E_{-2} = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \,, \qquad E_2 = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-3} = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \,, \qquad E_3 = \pm\sqrt{(1/6)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

The ambiguity in sign may be cleared up by employing the relation

$$[E_1, E_3] = N_{13}\, E_2 = +\sqrt{(1/6)}\;\; E_2$$

according to (4.10). This is satisfied by taking the plus sign in every case. One may verify that the commutation relations (3.13) are obeyed.

In the $D^{(3)}(0, 1)$ representation we assign to the weights $(-\frac{\sqrt{3}}{6}, -\frac{1}{6})$, $(\frac{\sqrt{3}}{6}, -\frac{1}{6})$, $(0, \frac{1}{3})$, the ket vectors

$$|\{3\}'', 1\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\{3\}'', 2\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$|\{3\}'', 3\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$ respectively. To avoid confusion we put two primes on the kets and Lie operators, and by our previous method obtain

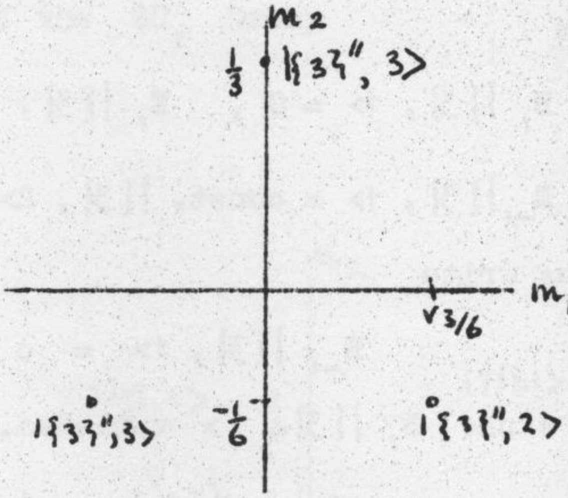


Fig. 27. The ket vectors for the $D^{(3)}(0,1)$ representation of SU_3.

$$H_1'' = -\frac{\sqrt{3}}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2'' = -\frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$E_1'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_3'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_{-1}'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-2}'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad E_{-3}'' = -\sqrt{\frac{1}{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that each matrix is minus the transpose of that representing the corresponding unprimed operator, and that every matrix is traceless.

The matrix representatives of the E-operators are clearly not hermitian but we can easily construct linear combinations of them that are hermitian and traceless, viz.

$$E_\alpha + E_{-\alpha}\ , \qquad -i\,(E_\alpha - E_{-\alpha})\ .$$

We shall henceforth understand by L_A the hermitian representatives of the operators of the SU_3 group. We shall still sometimes find it convenient to use primes for the operators of the $D^{(3)}(0,\ 1)$ representation, and we see that L_A'' is minus the transpose of L_A.

For the SU_2 group the infinitesimal transformation of an isospinor was

$$f' = (1 + i\,\varepsilon^A \tau_A)\, f\ ,$$

or more explicitly

$$f'_a = (\delta_a{}^b + i\,\varepsilon^A \tau_{Aa}{}^b)\, f_b\ ,$$

where τ_A is hermitian and traceless and ε^A is real. We now consider the transformation

$$\psi'_a = (\delta_a{}^b + i\,\varepsilon^A L_{Aa}{}^b)\,\psi_b\ , \tag{8.1}$$

where ε^A is real, $a = 1, 2, 3$ and A runs from 1 to 8. If we transform $\psi_a{}^*$, we get

$$\begin{aligned}\psi_a{}^{*\prime} &= (\delta_a{}^b - i\,\varepsilon^A L^*{}_{Aa}{}^b)\,\psi_b{}^* \\ &= (\delta^b{}_a - i\,\varepsilon^A L_{Ab}{}^a)\,\psi_b{}^*\ ,\end{aligned} \tag{8.2}$$

where we use the hermiticity of L_A. We had that

$$L_{Aa}{}^{\prime\prime b} = -L_{Ab}{}^a\ ,$$

so the previous equation may be expressed as

$$\psi_a{}^{*\prime} = (\delta_a{}^b + i\,\varepsilon^A L''{}_{Aa}{}^b)\,\psi_b{}^*\ ,$$

which is just (8.1) with L_A replaced by L''_A. Thus, when the Lie operators of $D^{(3)}(1,\ 0)$ give the infinitesimal transformation of ψ_a , the Lie operators of $D^{(3)}(0, 1)$ give the infinitesimal transformation of the conjugate complex $\psi_a{}^*$. For this reason we shall replace $|\{3\}'', a\rangle$ by $|\{3\}^*, a\rangle$. To a first order in the ε's equations (8.1) and (8.2) give

$$\begin{aligned}
\psi_a{}^{*\prime} \psi_a{}'' &= (\delta_a{}^b - i\,\varepsilon^A L_{Ab}{}^a)(\delta_a{}^c + i\,\varepsilon^B L_{Ba}{}^c)\psi_b{}^*\psi_c \\
&= \psi_a{}^*\psi_a + i\,\varepsilon^A (L_{Ab}{}^c - L_{Ab}{}^c)\psi_b{}^*\psi_c \\
&= \psi_a{}^*\psi_a\ ;
\end{aligned}$$

in other words, $\psi_1{}^*\psi_1 + \psi_2{}^*\psi_2 + \psi_3{}^*\psi_3$ is invariant under the infinitesimal transformations. Accordingly we have an infinitesimal unitary group in three dimensions. The determinant of the infinitesimal transformation matrix is

$$\begin{vmatrix}
1 + i\,\varepsilon^A L_{A1}{}^1 & i\,\varepsilon^A L_{A1}{}^2 & i\,\varepsilon^A L_{A1}{}^3 \\
i\,\varepsilon^A L_{A2}{}^1 & 1 + i\,\varepsilon^A L_{A2}{}^2 & i\,\varepsilon^A L_{A2}{}^3 \\
i\,\varepsilon^A L_{A3}{}^1 & i\,\varepsilon^A L_{A3}{}^2 & 1 + i\,\varepsilon^A L_{A3}{}^3
\end{vmatrix}$$

$$= 1 + i\,\varepsilon^A (L_{A1}{}^1 + L_{A2}{}^2 + L_{A3}{}^3) + \ldots$$

$$= 1 + i\,\varepsilon^A \operatorname{tr} L_A + \ldots = 1$$

to a first approximation, and the infinitesimal group is unimodular. By iteration we find that the finite transformations

$$\psi \to e^{i\theta^A L_A}\psi\,, \qquad \psi^* \to e^{-i\theta^A L_A}\psi^*$$

constitute a unimodular unitary group in three dimensions, which for this reason is called SU_3.

These considerations may serve as a preparation for the introduction of a tensor algebra for the three-dimensional representation of the group.

We define a <u>covariant vector</u> as a triad (ϕ_1, ϕ_2, ϕ_3) that transforms like (ψ_1, ψ_2, ψ_3) ; that is to say, the infinitesimal transformation is

$$\phi_a{}' = (\delta_a{}^b + i\,\varepsilon^A L_{Aa}{}^b)\,\phi_b .$$

We say that (χ^1, χ^2, χ^3) is a <u>contravariant vector</u>, if $\chi^a \phi_a$ is invariant. Then the infinitesimal transformation of χ^a must be

$$\chi^{a\prime} = (\delta_b{}^a - i\,\varepsilon^A L_{Ab}{}^a)\,\chi^b .$$

For the three-dimensional representation of SU_3 the conjugate complex of a covariant vector is a contravariant vector but this is not generally true for other Lie groups or for other representations of SU_3 . We shall often write $|\{3\}^*, a>$ as ψ^a .

When drawing weight diagrams we proved that $D^{(3)}(1, 0)$ and $D^{(3)}(0,1)$ are inequivalent representations. Thus it is not possible to express χ^a as a linear combination of the ϕ_a's and vice versa; there exists no metrical tensor h_{ab} that would enable us to raise and lower indices through a relation like $\phi_a = h_{ab}\,\chi^b$. While we have no metrical tensor, we do have invariants e.g. $\chi^a \phi_a$. Having defined covariant and contravariant vectors we can define a tensor $t^{abc\ldots}_{de\ldots}$ as a set of numbers that transform like $\psi^a \psi^b \psi^c \ldots \psi_d \psi_e \ldots$.

For future reference we write down the effect of operating with the H's and the E's on $|\{3\}, a>$ and $|\{3\}^*, a>$. We shall not use the two primes for H and E because there is no longer danger of ambiguity. On employing the explicit representations for the operators and the kets we see that the only non-vanishing results are

$$H_1|\{3\},1\rangle = \frac{\sqrt{3}}{6}|\{3\},1\rangle, \quad H_1|\{3\},2\rangle = -\frac{\sqrt{3}}{6}|\{3\},2\rangle$$

$$H_2|\{3\},1\rangle = \frac{1}{6}|\{3\},1\rangle, \quad H_2|\{3\},2\rangle = \frac{1}{6}|\{3\},2\rangle, \quad H_2|\{3\},3\rangle = -\frac{1}{3}|\{3\},3\rangle$$

$$E_1|\{3\},2\rangle = \frac{1}{\sqrt{6}}|\{3\},1\rangle, \quad E_2|\{3\},3\rangle = \frac{1}{\sqrt{6}}|\{3\},1\rangle, \quad E_3|\{3\},3\rangle = \frac{1}{\sqrt{6}}|\{3\},2\rangle$$

$$E_{-1}|\{3\},1\rangle = \frac{1}{\sqrt{6}}|\{3\},2\rangle, \quad E_{-2}|\{3\},1\rangle = \frac{1}{\sqrt{6}}|\{3\},3\rangle, \quad E_{-3}|\{3\},2\rangle = \frac{1}{\sqrt{6}}|\{3\},3\rangle$$

$$H_1|\{3\}^*,1\rangle = -\frac{\sqrt{3}}{6}|\{3\}^*,1\rangle, \quad H_1|\{3\}^*,2\rangle = \frac{\sqrt{3}}{6}|\{3\}^*,2\rangle$$

$$H_2|\{3\}^*,1\rangle = -\frac{1}{6}|\{3\}^*,1\rangle, \quad H_2|\{3\}^*,2\rangle = -\frac{1}{6}|\{3\}^*,2\rangle, \quad H_2|\{3\}^*,3\rangle = \frac{1}{3}|\{3\}^*,3\rangle$$

$$E_1|\{3\}^*,1\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,2\rangle, \quad E_2|\{3\}^*,1\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,3\rangle, \quad E_3|\{3\}^*,2\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,3\rangle$$

$$E_{-1}|\{3\}^*,2\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,1\rangle, \quad E_{-2}|\{3\}^*,3\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,1\rangle, \quad E_{-3}|\{3\}^*,3\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,2\rangle. \tag{8.4}$$

Eight-Dimensional Representation

Equation (7.1) shows that the product wave functions $|\{3\}^*,a\rangle|\{3\},b\rangle$ belong to $D^{(1)}(0, 0)$ or to $D^{(8)}(1, 1)$. We shall now face the problem of constructing the wave functions for the eight- and one-dimensional cases. We assign the ket vectors $|\{8\}, A\rangle$ to the weights of $D^{(8)}(1, 1)$ as shown in Fig. 28. Now Figs. 26 and 27 show that the weight of $|\{8\},1\rangle$ is the vectorial sum of the weights of $|\{3\},1\rangle$ and $|\{3\}^*,2\rangle$, so we put $|\{8\},1\rangle = |\{3\}^*,2\rangle|\{3\},1\rangle$. From these we construct the other $|\{8\},A\rangle$'s by successive applications of the displacement operators. If we operate with E_3 on $|\{8\},1\rangle$, we obtain

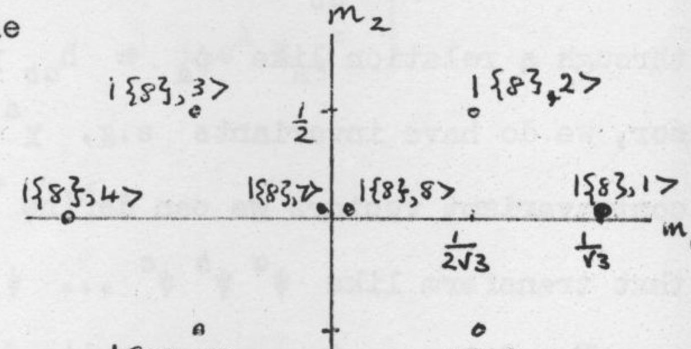


Fig. 28. The ket vectors for the $D^{(8)}(1,1)$ representation of SU_3.

a wave function with weight $(\frac{1}{\sqrt{3}}, 0) + \mathfrak{x}(3)$; that is to say, we have a multiple of $|\{8\},2\rangle$.

$$E_3|\{8\},1\rangle = E_3|\{3\}^*,2\rangle|\{3\},1\rangle$$

$$= E_3|\{3\}^*,2\rangle.|\{3\},1\rangle + |\{3\},2\rangle.E_3|\{3\},1\rangle$$

$$= -\frac{1}{\sqrt{6}}|\{3\}^*,3\rangle|\{3\},1\rangle$$

by (8.4) . In order to have $|\{8\},2\rangle$ normalized to unity we write

$$|\{8\},2\rangle = |\{3\}^*,3\rangle|\{3\},1\rangle ,$$

so that

$$E_3|\{8\},1\rangle = -\frac{1}{\sqrt{6}}|\{8\},2\rangle .$$

To obtain $|\{8\},3\rangle$ we operate with E_{-1} on $|\{8\},2\rangle$, which gives

$$E_{-1}|\{8\},2\rangle = \frac{1}{\sqrt{6}}|\{8\},3\rangle \text{ with } |\{8\},3\rangle = |\{3\}^*,3\rangle|\{3\},2\rangle .$$

Proceeding in this way we obtain

$$E_{-2}|\{8\},3\rangle = -\frac{1}{\sqrt{6}}|\{8\},4\rangle \text{ with } |\{8\},4\rangle = |\{3\}^*,1\rangle|\{3\},2\rangle$$

$$E_{-3}|\{8\},4\rangle = -\frac{1}{\sqrt{6}}|\{8\},5\rangle \text{ with } |\{8\},5\rangle = |\{3\}^*,1\rangle|\{3\},3\rangle$$

$$E_1|\{8\},5\rangle = -\frac{1}{\sqrt{6}}|\{8\},6\rangle \text{ with } |\{8\},6\rangle = |\{3\}^*,2\rangle|\{3\},3\rangle .$$

We see that any two kets are orthogonal, e.g.

$$\langle 4,\{8\}|\{8\},5\rangle = \langle 2,\{3\}\langle 1,\{3\}^*|\{3\}^*,1\rangle|\{3\},3\rangle$$

$$= \langle 2,\{3\}|\{3\},3\rangle = 0 .$$

This is as it should be, since they are members of a basic set of eigenvectors. We must next find the kets of zero weight.

$$E_1|\{8\},4\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,2\rangle|\{3\},2\rangle + \frac{1}{\sqrt{6}}|\{3\}^*,1\rangle|\{3\},1\rangle$$

$$E_2|\{8\},5\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,3\rangle|\{3\},3\rangle + \frac{1}{\sqrt{6}}|\{3\}^*,1\rangle|\{3\},1\rangle$$

$$E_3|\{8\},6\rangle = -\frac{1}{\sqrt{6}}|\{3\}^*,3\rangle|\{3\},3\rangle + \frac{1}{\sqrt{6}}|\{3\}^*,2\rangle|\{3\},2\rangle .$$

We have two, and only two, linearly independent kets. We write

$$|\{8\},7\rangle = \frac{1}{\sqrt{2}}\left(|\{3\}^*,1\rangle|\{3\},1\rangle - |\{3\}^*,2\rangle|\{3\},2\rangle\right),$$

the factor $\frac{1}{\sqrt{2}}$ being inserted for normalization, and

$$|\{8\},8\rangle = p\left(|\{3\}^*,2\rangle|\{3\},2\rangle - |\{3\}^*,3\rangle|\{3\},3\rangle\right) +$$
$$+ q\left(|\{3\}^*,3\rangle|\{3\},3\rangle - |\{3\}^*,1\rangle|\{3\},1\rangle\right).$$

The condition that this be orthogonal to $|\{8\},7\rangle$ gives $q = -p$ and the normalized ket is thus

$$|\{8\},8\rangle = \frac{1}{\sqrt{6}}\left(|\{3\}^*,1\rangle|\{3\},1\rangle + |\{3\}^*,2\rangle|\{3\},2\rangle - 2|\{3\}^*,3\rangle|\{3\},3\rangle\right).$$

We may establish an eight-dimensional representation of the Lie operators. We write

$$|\{8\},1\rangle = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \quad |\{8\},2\rangle = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \quad \ldots |\{8\},8\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}.$$

The H_1, H_2 diagonal elements are the m_1, m_2 values as shown on Figure 28 and can be written down immediately. To find the matrix representative of E_1 we see that it produces zero except in the following cases:

$$\begin{aligned} E_1|\{8\},3\rangle &= \frac{1}{\sqrt{6}}|\{8\},2\rangle, \quad E_1|\{8\},4\rangle = \frac{1}{\sqrt{3}}|\{8\},7\rangle \\ E_1|\{8\},5\rangle &= -\frac{1}{\sqrt{6}}|\{8\},6\rangle, \quad E_1|\{8\},7\rangle = -\frac{1}{\sqrt{3}}|\{8\},1\rangle. \end{aligned} \tag{8.5}$$

Thus the only non-vanishing elements of E_1 are

$$(E_1)_2{}^3 = \frac{1}{\sqrt{6}}, \quad (E_1)_7{}^4 = \frac{1}{\sqrt{3}}, \quad (E_1)_6{}^5 = -\frac{1}{\sqrt{6}}, \quad (E_1)_1{}^7 = -\frac{1}{\sqrt{3}}.$$

In this manner we obtain, with dots standing for zeros,

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad H_2 = \frac{1}{2}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$(8.6)$$

$$E_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot \\ \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad E_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & -\frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2\sqrt{3}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot \end{pmatrix},$$

$$E_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2\sqrt{3}} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2\sqrt{3}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot \end{pmatrix}$$

$$E_{-1} = E_1^{*}\ , \quad E_{-2} = E_2^{*}\ , \quad E_{-3} = E_3^{*}\ .$$

We observe that all the matrices are traceless. We may easily verify that the above matrices satisfy the commutation relations (3.13).

The E_1 , E_{-1} operators effect displacements between kets with the same m_2-eigenvalues. Equations (8.5) show that the states $|\{8\},3\rangle$ and $|\{8\},2\rangle$ constitute an m_1-doublet, that $|\{8\},4\rangle$, $|\{8\},7\rangle$ and $|\{8\},1\rangle$ form a triplet, that $|\{8\},5\rangle$ and $|\{8\},6\rangle$ form a doublet, and that $|\{8\},8\rangle$ is an m_1-singlet. To put this differently : H_1 , E_1 , E_{-1} are the operators of an SU_2 group which is a subgroup of SU_3 . They are not, however, elements of an invariant subalgebra because, for example, $[E_1, E_3]$ is not a linear combination of H_1 , E_1 , E_{-1} .

We write ϕ_A for $|\{8\},A\rangle$ and define a covariant vector in eight dimensions as a set of eight numbers that transform like ϕ_A . We have

$$\phi_A = D_{Aa}^{\ \ b}\,|\{3\}^*,a\rangle|\{3\}b\rangle = D_{Aa}^{\ \ b}\,\psi^a\,\psi_b\ , \qquad (8.7)$$

where

$$D_{1a}^{\ \ b} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ D_{2a}^{\ \ b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ D_{3a}^{\ \ b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ D_{4a}^{\ \ b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_{5a}^{\ \ b} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad D_{6a}^{\ \ b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\quad D_{7a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_{8a}^{\ \ b} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

according to the explicit expressions for ϕ_A . Making infinitesimal transformations of ψ^a and ψ_b in (8.7) we obtain

$$\phi_A' = D_{Aa}{}^{b}\,(\delta_c{}^{a} - i\,\varepsilon^F L_{Fc}{}^{a})\,(\delta_b{}^{a} + i\,\varepsilon^G L_{Gb}{}^{d})\,\psi^c\,\psi_d$$
$$= \phi_A + i\,\varepsilon^F\,(L_{Fb}{}^{d}\,D_{Ac}{}^{b} - L_{Fc}{}^{a}\,D_{Aa}{}^{d})\,\psi^c\,\psi_d\,.$$

Since the ϕ_A's form a complete set of eigenvectors, they transform among themselves and it is therefore possible to express the last equation as

$$\phi_A' = (\delta_A{}^{B} + i\,\varepsilon^F \mathcal{L}_{FA}{}^{B})\,\phi_B\,, \tag{8.8}$$

where ε^F is real and $\mathcal{L}_{FA}{}^{B}$ is eight-dimensional. This relation gives the infinitesimal transformation of a covariant vector. We define a contravariant eight-vector χ^A as a set of eight numbers such that $\chi^A\,\phi_A$ is invariant under the group transformations.

Let the infinitesimal transformation of χ^A be

$$\chi'^{A} = (\delta_B{}^{A} + i\,\varepsilon^F\,\mathcal{L}''_{FB}{}^{A})\,\chi^B\,. \tag{8.9}$$

Then

$$\chi'_A\,\phi'_A = (\delta^A{}_B + i\,\varepsilon^F \mathcal{L}''_{FB}{}^{A})\,(\delta_A{}^{C} + i\,\varepsilon^G \mathcal{L}_{GA}{}^{C})\,\chi^B\,\chi_C$$
$$= \chi^A\,\phi_A + i\,\varepsilon^F\,(\mathcal{L}''_{FB}{}^{C} + \mathcal{L}_{FB}{}^{C})\,\chi^B\,\chi_C$$

and for $\chi^A\,\phi_A$ to be invariant we must have

$$\mathcal{L}''_{FB}{}^{C} = -\,\mathcal{L}_{FB}{}^{C}\,,$$

so that (8.9) is expressible as

$$\chi'^{A} = (\delta^A{}_B - i\,\varepsilon^F \mathcal{L}_{FB}{}^{A})\,\chi^B\,.$$

Having defined covariant and contravariant vectors by their infinitesimal transformations we can, as in the three-dimensional case, define a tensor.

Equation (8.7) shows that $D_{Aa}{}^{b}$ is eight-dimensional covariant with respect to A , three-dimensional covariant with respect to a , and three-dimensional contravariant with respect to b . Hence

$$g_{AB} = D_{Aa}{}^{b}\,D_{Bb}{}^{a} = \mathrm{tr}\,D_A\,D_B$$

is a covariant tensor of the second rank in eight dimensions, which we may call the metrical tensor. It is symmetric and the explicit expressions for the D's show that its non-vanishing elements are

$$g_{14}\ ,\quad g_{25}\ ,\quad g_{36}\ ,\quad g_{41}\ ,\quad g_{52}\ ,\quad g_{63}\ ,\quad g_{77}\ ,\quad g_{88}$$

and that they are all equal to unity. It is not difficult to verify that

$$g_{AB} = C_{AD}{}^{E}\, C_{BE}{}^{D}\ ,$$

where the C's are the structure constants corresponding to the Lie operators **D**. The determinant of g_{AB} is equal to unity and one finds that the elements of the associated tensor g^{AB} defined by

$$g^{AB}\, g_{BF} = \delta^{A}{}_{F}$$

have the same numerical values as g_{AB}, so that

$$g_{AB} = g_{BA} = g^{AB} = g^{BA} = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} .$$

With every covariant χ_A we can associate a contravariant χ^A through the relation

$$\chi^{A} = g^{AB}\,\chi_{B} \tag{8.10}$$

$$\chi^{1} = \chi_{4}\ ,\ \chi^{2} = \chi_{5}\ ,\ \chi^{3} = \chi_{6}\ ,\ \chi^{4} = \chi_{1}\ ,\ \chi^{5} = \chi_{2}\ ,\ \chi^{6} = \chi_{3}\ ,\ \chi^{7} = \chi_{7}\ ,$$

$$\chi^{8} = \chi_{8}\ .$$

In particular the ϕ^A 's are situated on the weight diagram as shown in Figure 29; the weight of ϕ^A is the negative of the weight of ϕ_A and, if ϕ_A is the ket vector of a particle state, ϕ^A is the ket vector of the antiparticle state. On account of (8.10) the representations with basis vectors ϕ_A and ϕ^A are equivalent. Moreover the weight diagram of one is obtained from that of the other by reflection through the origin. We denote this reflection by R . It is not an operation of the group because it is not expressible as a linear combination of the E_a 's .

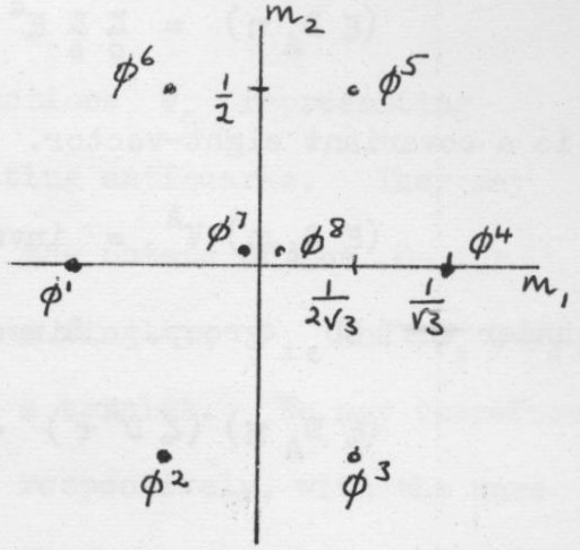


Fig. 29. The contravariant ket vector for the $D^8(1,1)$ representation of SU_3 .

We can also associate eight-dimensional operators O_A , O^A through the relation

$$O^A = g^{AB} O_B .$$

The O_B may be the D 's and in the equation

$$D^A = g^{AB} D_B$$

we understand that we take the same a , b indices for D^A and D_B , so that

$$D^1{}_a{}^b = D_{4a}{}^b , \quad D^2{}_a{}^b = D_{5a}{}^b , \quad D^3{}_a{}^b = D_{6a}{}^b , \quad D^4{}_a{}^b = D_{1a}{}^b$$

$$D^5{}_a{}^b = D_{2a}{}^b , \quad D^6{}_a{}^b = D_{3a}{}^b , \quad D^7{}_a{}^b = D_{7a}{}^b , \quad D^8{}_a{}^b = D_{8a}{}^b .$$

If ξ^a is a contravariant three-vector and η_b is any covariant three-vector, then $(\xi D_A \eta)$ defined by

$$(\xi\, D_A\, \eta) = \sum_a \sum_b \xi^a\, D_{A_a}{}^b\, \eta_b$$

is a covariant eight-vector. Hence for any contravariant eight-vector V^A

$$(\xi\, D_A\, \eta)\, V^A = \text{invariant}$$

under the SU_3 group. Since $(\zeta\, D^A\, \tau)$ is a contravariant eight-vector,

$$(\xi\, D_A\, \eta)\, (\zeta\, D^A\, \tau) = \text{invariant.} \tag{8.11}$$

<u>One-Dimensional Representation</u>

We lastly turn to the one-dimensional representation of the group. It is contained in the product $D^{(3)}(0,\, 1) \otimes D^{(3)}(1,\, 0)$, so the singlet ket vector is the sum of products of $|\{3\}^*, a\rangle$ and $|\{3\}, b\rangle$. In order to have zero weight it must be of the form

$$c_1 |\{3\}^*, 1\rangle |\{3\}, 1\rangle + c_2 |\{3\}^*, 2\rangle |\{3\}, 2\rangle + c_3 |\{3\}^*, 3\rangle |\{3\}, 3\rangle\ . \tag{8.12}$$

Since it is the only member of the set, the displacement operators must annihilate it and this gives $c_1 = c_2 = c_3$. The normalized ket is then

$$|\{1\}, 1\rangle = \frac{1}{\sqrt{3}} \left(|\{3\}^*, 1\rangle |\{3\}, 1\rangle + |\{3\}^*, 2\rangle |\{3\}, 2\rangle + |\{3\}^*, 3\rangle |\{3\}, 3\rangle \right)\ .$$

This we knew already. If (8.12) is annihilated by the displacement operators E_α and the weight operators H_i, the L_A in the infinitesimal operation $1 + i\, \varepsilon^A L_A$ is identically zero. It follows that the the infinitesimal transformation is the identity transformation so that (8.12) is invariant under SU_3. It is therefore a constant times $\psi^a \psi_a$ or $\sum_{a=1}^{3} |\{3\}^*, a\rangle |\{3\}, a\rangle$. A one-dimensional invariant equation similar to (8.11) is

$$(\xi^a\, \delta_a{}^b\, \eta_b)\, (\zeta^c\, \delta_c{}^d\, \tau_d) = \text{invariant.}$$

Ket Vectors for Mesons and Baryons

The ϕ_A are sums of products of wave functions ψ_a representing quarks and ψ^a with opposite weights representing antiquarks. They may therefore be interpreted as the ket vectors of the octets of the 0- or the 1- bosons. Moreover we saw that ϕ_8 is a singlet, ϕ_3 and ϕ_2, ϕ_5 and ϕ_6 are doublets, and ϕ_4, ϕ_7, ϕ_1 is a triplet. We may therefore identify $D_{Aa}{}^b \psi^a \psi_b$ with A = 1, 2, ..., 8, respectively, with the wave functions of the pseudoscalar mesons

$$\pi^+ , \; K^+ , \; K^o , \; \pi^- , \; K^- , \; \overline{K^o} , \; \pi^o , \; \eta$$

and with the wave functions of the vector mesons

$$\rho^+ , \; K^{*+} , \; K^{*o} , \; \rho^- , \; \overline{K^{*+}} , \; \overline{K^{*o}} , \; \rho^o , \; (\omega^o , \phi^o) ,$$

where (ω^o , ϕ^o) stands for the combination of ω^o and ϕ^o that belongs to the octet. The singlet member of the nonet will have the wave function $|\{1\},1\rangle$, that is, $\frac{1}{\sqrt{3}} \psi^a \psi_a$.

The baryon wave functions are sums of products of three quark wave functions that constitute an eight-vector. By drawing weight diagrams we can easily see that

$$D^{(3)}(1, 0) \otimes D^{(3)}(1, 0) = D^{(6)}(2, 0) \oplus D^{(3)}(0, 1) .$$

The $D^{(3)}(0, 1)$ ket vectors are readily found to be

$$u \equiv \frac{1}{\sqrt{2}} (\psi_2 \psi_3 - \psi_3 \psi_2), \quad v \equiv \frac{1}{\sqrt{2}} (\psi_3 \psi_1 - \psi_1 \psi_3) , \quad w \equiv \frac{1}{\sqrt{2}} (\psi_1 \psi_2 - \psi_2 \psi_1)$$

On applying E_1, E_2, E_3 we obtain the only non-vanishing results

$$E_1 u = -\sqrt{\frac{1}{6}}\, v , \quad E_2 u = -\sqrt{\frac{1}{6}}\, w , \quad E_3 v = -\sqrt{\frac{1}{6}}\, w .$$

Thus (u, v, w) have the same weights and obey the same equations as $(\psi^1 , \psi^2 , \psi^3)$. We may therefore identify the two triads. Employing the totally antisymmetric Levi Civita symbol ε^{abc} which has the value unity for $a = 1$, $b = 2$, $c = 3$ we write

$$\psi^a = \frac{1}{2\sqrt{2}} \varepsilon^{abc} (\psi_b \psi_c - \psi_c \psi_b)$$

$$= \frac{1}{\sqrt{2}} \varepsilon^{abc} \psi_b \psi_c$$

While there is no linear relation between ψ^a and the ψ_b's , there is such a relation between ψ^a and the products of ψ_b and ψ_c . The eight-dimensional vector may be written as

$$\phi_A = D_{Aa}{}^d \psi^a \psi_d = \frac{1}{\sqrt{2}} D_{Aa}{}^d \varepsilon^{abc} \psi_b \psi_c \psi_d .$$

This is the sum of products of three quark wave functions and for A running from 1 to 8 we may identify ϕ_A expressed in this form with the ket vectors of the baryons

$$\Sigma^+ ,\ p ,\ n ,\ \Sigma^- ,\ \Xi^- ,\ \Xi^0 ,\ \Sigma^0 ,\ \Lambda^0 .$$

CHAPTER IX:

The B_2 Group

Representations of the B_2 Group.

We base the investigation of the B_2 group on the four-dimensional representation, associating the fundamental covariant four-vectors

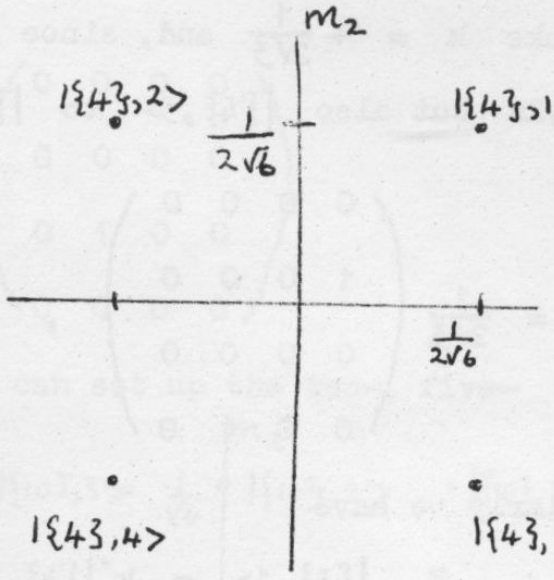


Fig. 30. The ket vectors for the $D^{(4)}(1,0)$ representation of B_2.

$$\psi_1 \equiv |\{4\},1> \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_2 \equiv |\{4\},2> \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_3 \equiv |\{4\},3> \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi_4 \equiv |\{4\},4> \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with the weights

$$\left(\frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{6}}\right), \quad \left(-\frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{6}}\right), \quad \left(\frac{1}{2\sqrt{6}}, -\frac{1}{2\sqrt{6}}\right), \quad \left(-\frac{1}{2\sqrt{6}}, -\frac{1}{2\sqrt{6}}\right),$$

respectively. Thus we represent H_1 and H_2 as

$$H_1 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

On referring to Figure 7 we see that $E_1|\{4\},1>$ vanishes and, if we write

$$E_{-1}|\{4\},1> = k|\{4\},2>,$$

we deduce from (5.2) that

$$k^2 = (\underline{r}(1)\cdot\underline{m}(1)) = \frac{1}{12}\,.$$

We take $k = +\frac{1}{2\sqrt{3}}$ and, since E_{-1} shifts not only $|\{4\},1\rangle$ to $|\{4\},2\rangle$ but also $|\{4\},3\rangle$ to $|\{4\},4\rangle$, we write

$$E_{-1} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad E_{1} = E_{-1}^{+} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly we have

$$E_{-2}\,|\{4\},1\rangle = k'|\{4\},4\rangle\,,$$

where

$$k'^2 = (\underline{r}(2)\cdot\underline{m}(1)) = 1/6\,.$$

We take k' equal to $+1/\sqrt{6}$ and write

$$E_{-2} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad E_{2} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From now on we must be careful about phase factors. To calculate E_3 we employ the relation

$$[E_2, E_{-1}] = N_{2,-1}\,E_3\,,$$

where the value $-\sqrt{\frac{1}{6}}$ of $N_{2,-1}$ is taken from (4.10), and find

$$E_{3} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{-3} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Finally we deduce from the relation

$$[E_3, E_{-1}] = N_{3,-1}\ E_4 = \frac{1}{\sqrt{6}} E_4$$

that

$$E_4 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{-4} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From products of two four-dimensional kets we can set up the ten-, five- and one-dimensional representations.

We assign the kets for the regular representation $D^{(10)}(2,0)$ as shown in Figure 31. In the product representation the point with weight $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ belongs only to the ten-dimensional representation and its ket vector must be $|\{4\},1\rangle|\{4\},1\rangle$ in order to have the correct weight. By applying the displacement operators we find the orthonormal set of kets as follows:

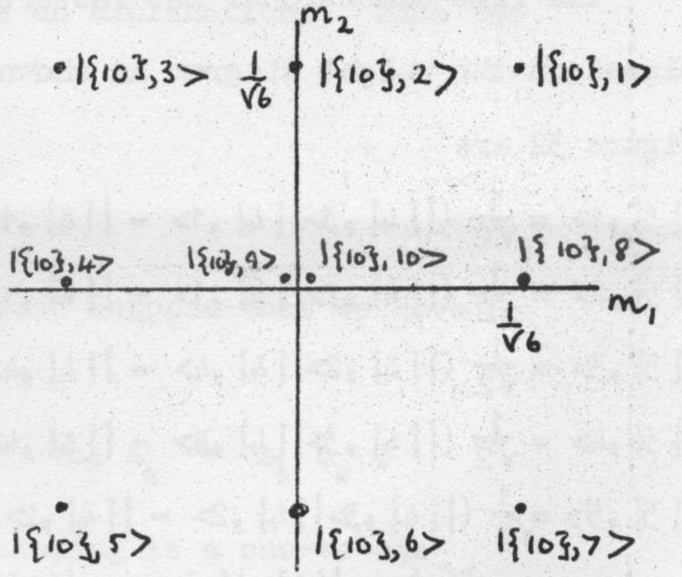


Fig. 31. The ket vectors for the $D^{(10)}(2,0)$ representation of B_2.

$$|\{10\},1\rangle = |\{4\},1\rangle|\{4\},1\rangle, \qquad |\{10\},2\rangle = \frac{1}{\sqrt{2}}\left(|\{4\},1\rangle|\{4\},2\rangle + |\{4\},2\rangle|\{4\},1\rangle\right)$$

$$|\{10\},3\rangle = |\{4\},2\rangle|\{4\},2\rangle, \qquad |\{10\},4\rangle = \frac{1}{\sqrt{2}}\left(|\{4\},2\rangle|\{4\},4\rangle + |\{4\},4\rangle|\{4\},2\rangle\right)$$

$$|\{10\},5\rangle = |\{4\},4\rangle|\{4\},4\rangle, \qquad |\{10\},6\rangle = \frac{1}{\sqrt{2}}\left(|\{4\},3\rangle|\{4\},4\rangle + |\{4\},4\rangle|\{4\},3\rangle\right)$$

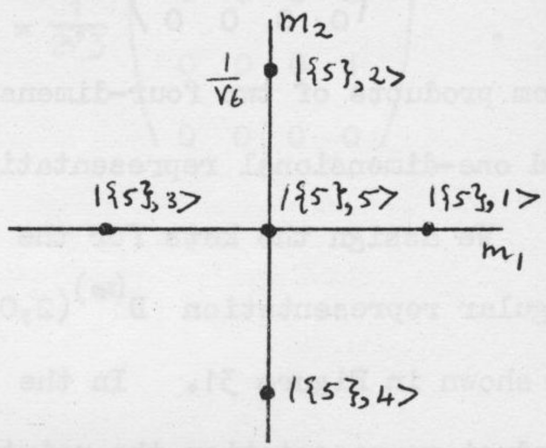
$$|\{10\},7\rangle = |\{4\},3\rangle|\{4\},3\rangle, \qquad |\{10\},8\rangle = -\frac{1}{\sqrt{2}}\left(|\{4\},1\rangle|\{4\},3\rangle + |\{4\},3\rangle|\{4\},1\rangle\right)$$

$$|\{10\},9\rangle = \frac{1}{2}\left(|\{4\},2\rangle|\{4\},3\rangle + |\{4\},3\rangle|\{4\}\,2\rangle + |\{4\},1\rangle|\{4\},4\rangle + |\{4\},4\rangle|\{4\},1\rangle\right)$$

$$|\{10\},10\rangle = \frac{1}{2}\left(|\{4\},2\rangle|\{4\},3\rangle + |\{4\},3\rangle|\{4\},2\rangle - |\{4\},1\rangle|\{4\},4\rangle - |\{4\},4\rangle|\{4\},1\rangle\right). \tag{9.1}$$

The $|\{10\},9\rangle$ and $|\{10\},10\rangle$ are linearly independent mutually orthogonal and of weight zero. The first is obtained by displacing $|\{10\},4\rangle$ along the m_1-axis and the second by displacing $|\{10\},6\rangle$ along the m_2-axis. If we had displaced $|\{10\},5\rangle$ to the origin, we would have obtained $\frac{1}{\sqrt{2}}(|\{10\},9\rangle - |\{10\},10\rangle)$ and, if we had displaced $|\{10\},7\rangle$, we would have obtained $\frac{1}{\sqrt{2}}(|\{10\},9\rangle + |\{10\},10\rangle)$. Thus we have only two independent kets at the origin.

The five-dimensional ket vectors placed on the weight diagram as shown in Figure 32 are



$$
\begin{aligned}
|\{5\},1\rangle &= \frac{1}{\sqrt{2}}(|\{4\},3\rangle|\{4\},1\rangle - |\{4\},1\rangle|\{4\},3\rangle) \\
|\{5\},2\rangle &= \frac{1}{\sqrt{2}}(|\{4\},2\rangle|\{4\},1\rangle - |\{4\},1\rangle|\{4\},2\rangle) \\
|\{5\},3\rangle &= \frac{1}{\sqrt{2}}(|\{4\},2\rangle|\{4\},4\rangle - |\{4\},4\rangle|\{4\},2\rangle) \\
|\{5\},4\rangle &= \frac{1}{\sqrt{2}}(|\{4\},3\rangle|\{4\},4\rangle - |\{4\},4\rangle|\{4\},3\rangle) \qquad (9.2) \\
|\{5\},5\rangle &= \frac{1}{\sqrt{2}}(|\{4\},3\rangle|\{4\},2\rangle - |\{4\},2\rangle|\{4\},3\rangle + \\
&\quad + |\{4\},4\rangle|\{4\}\rangle|\{4\},1\rangle - |\{4\},1\rangle|\{4\},4\rangle) .
\end{aligned}
$$

Fig. 32. The ket vectors for the $D^{(5)}(0,1)$ representation of B_2.

The expression for $|\{5\},1\rangle$ can contain only the products $|\{4\},1\rangle|\{4\},3\rangle$ and $|\{4\},3\rangle|\{4\},1\rangle$ in order to have the correct weight. We write $|\{5\},1\rangle$ as above in order to have it orthogonal to $|\{10\},8\rangle$. The other kets result from displacing. Every ket is clearly orthogonal to every ten-dimensional ket.

The one-dimensional ket vector has zero weight and it must vanish when the displacement operators are applied. This gives

$$
|\{1\},1\rangle = \frac{1}{2}(|\{4\},2\rangle|\{4\},3\rangle - |\{4\},3\rangle|\{4\},2\rangle + |\{4\},4\rangle|\{4\},1\rangle - |\{4\},1\rangle|\{4\},4\rangle), \qquad (9.3)
$$

which means that the sum of products

$$- x_1 y_4 + x_4 y_1 - x_3 y_2 + x_2 y_3 \qquad (9.4)$$

is invariant under B_2. Now any skew symmetric form

$$\sum_{i,k=1}^{4} a_{ik} x_i y_k \qquad (a_{ki} = - a_{ik}) \qquad (9.5)$$

can be reduced by a linear transformation to (9.4).[1] The group that leaves (9.5) invariant is called the two-dimensional <u>symplectic group</u> and is denoted by Sp_2 or C_2. What we have shown is that B_2 is <u>isomorphic</u> to C_2. This term requires an explanation. Take two groups A, B with elements

$$\begin{array}{lllll} A & a_0 & a_1 & a_2 & \ldots \\ B & b_0 & b_1 & b_2 & \ldots \end{array}$$

a_0 and b_0 being the identity elements. Suppose that we have a mapping of A onto B, that is,

$$a_0 \rightarrow b_0 , \qquad a_i \rightarrow b_i , \qquad a_i a_k \rightarrow b_i b_k .$$

Then B is homomorphic to A. If the mapping is a one-to-one correspondence, B is isomorphic to A.

<u>Tensor Algebra for B_2</u>.

A tensor algebra may be set up for $D^{(4)}(1,0)$ as was done in the case of the SU_3 group. We define a covariant four-vector as a set of four numbers (f_1, f_2, f_3, f_4) that transform like $(\psi_1, \psi_2, \psi_3, \psi_4)$. From the E-operators and H_1, H_2 we form ten independent, hermitian and traceless $L_{Aa}{}^{b}$ and make the infinitesimal transformation

$$f_a{}' = (\delta_a{}^b + i \varepsilon^A L_{Aa}{}^b) f_b , \qquad (9.6)$$

where ε^A is real, $A = 1, 2, \ldots, 10$; $a, b = 1, 2, 3, 4$. Then a

---

(1) H. Weyl, <u>The Theory of Groups and Quantum Mechanics</u>, p. 397 (Dover, 1930).

contravariant vector g^a is defined by the property that $g^a f_a$ is invariant, and its infinitesimal transformation is therefore

$$g^{a\prime} = (\delta^a_{\ b} - i\varepsilon^A L_{Ab}^{\ \ a})\, g^b . \tag{9.7}$$

We deduce from (9.6) and (9.7) that the diagonal weight operators for ψ^a are the negatives of those for ψ_a, so that the weight of ψ^a is minus the weight of ψ_a. We see that $f_a{}^*$ transforms like g^a and so $f_a{}^* f_a$ is invariant, but we are not working in the SU_4 group. In fact for this group we would have to make use of all the unimodular four-by-four unitary matrices. There are $4^2 - 1 = 15$ of these and only ten of them belong to B_2. However, B_2 is a subgroup of SU_4.

According to (9.3) we have an invariant

$$\psi_4 \psi_1 - \psi_3 \psi_2 + \psi_2 \psi_3 - \psi_1 \psi_4 .$$

This leads us to associate with ψ_a a contravariant $\psi^a = h^{ab}\psi_b$, where

$$h^{ab} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

is an antisymmetric contravariant tensor of the second rank. We can define an associated covariant tensor of the second rank h_{ab} by the relation

$$h_{ab}\, h^{bc} = \delta_a^{\ c}$$

and it will be seen immediately that $h_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Moreover

$$\psi_a = \delta_a^{\ c}\psi_c = h_{ab}\, h^{bc}\psi_c = h_{ab}\,\psi^b$$

and we may regard h_{ab} as a metrical tensor. On transforming under the group $\psi^{a}\psi_{a}$ becomes $\psi'^{a}\psi'_{a}$; that is to say,

$$h'^{ab}\psi'_{b}\psi'_{a} = h^{ab}\psi_{b}\psi_{a} ,$$

where h'^{ab} is a tensor identical with h^{ab} . A tensor whose elements have the same numerical values in the two systems is said to be <u>form invariant</u>. Thus h^{ab} and consequently h_{ab} are form invariant.

To investigate the tensor properties of the five-dimensional representation we write

$$|\{5\},i\rangle = \sigma_{i}^{\ ab}\psi_{a}\psi_{b}$$

and the elements of the five four-by-four matrices $\sigma_{i}^{\ ab}$ may be read off from (9.2). We can form the mixed tensor $\sigma_{ia}^{\ \ b}$, where a labels rows and b columns, from

$$\sigma_{ia}^{\ \ b} = h_{ac}\sigma_{i}^{\ cb} = (h\sigma_{i})_{a}^{\ b}$$

with the values

$$\sigma_{1a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sigma_{2a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\sigma_{3a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{4a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_{5a}^{\ \ b} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (9.8)$$

We define a covariant five-vector as a set of five numbers that transform like

$$|\{5\},1\rangle\ , \quad |\{5\},2\rangle\ , \quad |\{5\},3\rangle\ , \quad |\{5\},4\rangle\ , \quad |\{5\},5\rangle\ .$$

These we write ϕ_1 , ϕ_2 , ϕ_3 , ϕ_4 , ϕ_5 , and we see that

$$\phi_i = \sigma_i^{\ ab}\ \psi_a\,\psi_b = h_{ad}\ h^{dc}\ \psi_c\ \sigma_i^{\ ab}\ \psi_b$$

$$= -\psi^d\ \sigma_{id}^{\ \ b}\ \psi_b = -(\psi\,\sigma_i\,\psi)\ ,$$

say. It is clear that the vector property holds for $(\xi\,\sigma_i\,\psi)$, where ξ is a contravariant four-vector not associated with ψ , i.e. $\xi^d \neq h^{dc}\ \psi_c$. As in the case of SU_3 we can construct matrices $\mathcal{L}_{Ai}^{\ \ j}$ which give an infinitesimal transformation

$$\phi_i' = (\delta_i^{\ j} + i\ \varepsilon^A \mathcal{L}_{Ai}^{\ \ j})\ \phi_j\ . \qquad (A = 1,\ 2,\ \ldots\ 10;\ \ i,\ j = 1,2,3,4)$$

Then a contravariant vector Λ^i is one such that $\Lambda^i\ \phi_i$ is invariant. Its infinitesimal transformation is

$$\Lambda^{i\,\prime} = (\delta_j^{\ i} - i\ \varepsilon^A \mathcal{L}_{Aj}^{\ \ i})\ \phi^j\ .$$

A metrical tensor in five dimensions g_{ij} may be defined by

$$g_{ij} = \sigma_{ia}^{\ \ b}\ \sigma_{jb}^{\ \ a} = \mathrm{tr}\,\sigma_i\,\sigma_j\ ,$$

and an associated contravariant g^{ij} by

$$g^{il}\ g_{lj} = \delta_j^{\ i}\ .$$

Equations (9.6) show that

$$g_{ij} = g_{ji} = g^{ij} = g^{ji} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

We can then associate vectors and operators as follows:

$$\phi^1 = \phi_3\ ,\quad \phi^2 = -\phi_4,\quad \phi^3 = \phi_1,\quad \phi^4 = -\phi_2,\quad \phi^5 = \phi_5\ ,$$
$$\sigma^1 = \sigma_3\ ,\quad \sigma^2 = -\sigma_4,\quad \sigma^3 = \sigma_1,\quad \sigma^4 = -\sigma_2,\quad \sigma^5 = \sigma_5\ , \tag{9.9}$$

the a , b indices being put in the same positions for the σ -matrices. Since $(\xi\,\sigma_i\,\psi)$ transforms like ϕ_i , the expression $(\xi\,\sigma_i\,\psi)\,A^i$ is invariant for any contravariant five-vector A^i . We may, for example, put A^i equal to $(\eta\,\sigma^i\,\zeta)$ and obtain

$$(\xi\,\sigma_i\,\psi)\,(\eta\,\sigma^i\,\zeta) = \text{invariant}\ . \tag{9.10}$$

In ten dimensions one works in a similar manner. We write the fundamental covariant vector

$$\chi_A = |\{10\}, A> = T_A^{\ ab}\,\psi_a\,\psi_b\ ,$$

the $T_A^{\ ab}$ being given in (9.1). We evaluate $T_{Aa}^{\ \ b} = h_{ae}\,T_A^{\ eb}$ finding

$$T_{1a}^{\ \ b} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix},\quad T_{2a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&1&0&0 \end{pmatrix}$$

$$T_{3a}^{\ \ b} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix},\quad T_{4a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0 \end{pmatrix},$$

$$T_{5a}^{\ \ b} = \begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},\quad T_{6a}^{\ \ b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&-1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$T_{7a}{}^{b} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_{8a}{}^{b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$T_{9a}{}^{b} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{10a}{}^{b} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can construct a metrical tensor γ_{AB} from

$$\gamma_{AB} = \mathrm{tr}\, T_A T_B .$$

This gives associated covariant and contravariant ten-vectors

$$\chi^1 = -\chi_5, \quad \chi^2 = \chi_6, \quad \chi^3 = -\chi_7, \quad \chi^4 = \chi_8, \quad \chi^5 = -\chi_1,$$

$$\chi^6 = \chi_2, \quad \chi^7 = -\chi_3, \quad \chi^8 = \chi_4, \quad \chi^9 = \chi_9, \quad \chi^{10} = \chi_{10},$$

with similar relations between covariant T_A and contravariant T^A. Finally $(\xi\, T_A\, \psi) \equiv \xi^a\, T_{Aa}{}^{b}\, \psi^b$ is a covariant vector and

$$(\xi\, T_A\, \psi)\,(\eta\, T^A\, \zeta) = \text{invariant}. \qquad (9.11)$$

<u>Application to Leptonic Processes</u>

We shall now show how conservation laws for leptonic processes are related to invariances of the interaction Lagrangian density under the B_2 group transformations(1). According to Figure 3 we associate the four-dimensional kets $\psi_1, \psi_2, \psi_3, \psi_4$ with $\nu_\mu, \nu_e, \mu^-, e^-$, respectively. For their weak interaction we adopt the V-A theory of

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(1) J. McConnell, <u>Canadian Journal of Physics</u> <u>43</u>, 705 (1965), ~~in press~~

Feynman and Gell-Mann[1] modified by the presence of distinct electron and muon neutrinos. The interaction Lagrangian density is

$$\mathcal{L}(x) = \frac{G}{\sqrt{2}} J_\alpha^{+}(x) \mathcal{J}_\alpha(x) , \tag{9.12}$$

where

$$\mathcal{J}_\alpha(x) = i\,(\bar{\nu}_e(x)\,\gamma_\alpha(1+\gamma_5)\,e(x)) + i\,(\bar{\nu}_\mu(x)\,\gamma_\alpha(1+\gamma_5)\,\mu(x)) . \tag{9.13}$$

In (9.13) $a(x)$ denotes an operator that annihilates the particle a or creates its antiparticle $\bar{a}$, $a^+(x)$ the hermitian conjugate of $a(x)$ is an operator that annihilates $\bar{a}$ or creates a , and $\bar{a}(x) = a^+(x)\,\gamma_4$. The γ's are Dirac matrices and we work in units such that $\hbar = c = 1$. The reactions given by (9.12) clearly satisfy conservation of the lepton number, muon number and charge. The part of $\mathcal{L}(x)$ for any of the processes

$$\begin{array}{ll} f_1 + f_4 \to f_2 + f_3 , & f_1 + \bar{f}_2 \to f_3 + \bar{f}_4 \\ f_1 + \bar{f}_3 \to f_2 + \bar{f}_4 , & f_1 \to f_2 + f_3 + \bar{f}_4 , \text{ etc.} \end{array} \tag{9.14}$$

is

$$\frac{G}{\sqrt{2}} (\bar{\psi}_{f_2}\, \gamma_\alpha (1+\gamma_5)\, \psi_{f_1}) (\bar{\psi}_{f_3}\, \gamma_\alpha (1+\gamma_5)\, \psi_{f_4}) . \tag{9.15}$$

We have to introduce elements from the B_2 group. We recall that $\psi_a{}^*$ transforms like the contravariant ψ^a and the same is obviously true for $\overline{\psi_a}$. The wave functions in (9.15) now contain the specifications of the positions of the particles in the $D^{(4)}(1,0)$ weight diagram and on omitting space and spin dependences (9.15) has the general form

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(1) R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

$$(\psi^{f_2} A \psi_{f_1})(\psi^{f_3} B \psi_{f_4}) , \tag{9.16}$$

where A and B are group operators. According to (9.10) and (9.11) there are two invariants

$$\sum_{a,b,c,d,i} (\psi^a \sigma_{ia}{}^b \psi_b)(\psi^c \sigma^i{}_c{}^d \psi_d) , \qquad \sum_{a,b,c,d,i} (\psi^a T_{Aa}{}^b \psi_b)(\psi^c T^A{}_c{}^d \psi_d) .$$

On account of the invariance of $\psi^a \psi_a$ we have another invariant

$$\sum_{a,b,c,d} (\psi^a \delta_a{}^b \psi_b)(\psi^c \delta_c{}^d \psi_d) .$$

If we want a four fermion interaction Lagrangian density that is invariant under the group transformations, it must assume the form

$$a_1 (\psi^a \delta_a{}^b \psi_b)(\psi^c \delta_c{}^d \psi_d) + a_5 (\psi^a \sigma_{ia}{}^b \psi_b)(\psi^c \sigma^i{}_c{}^d \psi_d) + a_{10} (\psi^a T_{Aa}{}^b \psi_b)(\psi^c T^A{}_c{}^d \psi_d) , \tag{9.17}$$

summed over the repeated indices. The a_1, a_5, a_{10} contain the space and spin dependent parts but they are independent of each other. Each product of two brackets is an invariant function and so is annihilated by a hermitian Lie operator L . Since two of these operators are H_1 and H_2 , each product has zero weight. We therefore have conservation of charge and of muon number.

To see this more clearly we take the five-dimensional representation. According to (9.9)

$$\sigma^1{}_a{}^b = \sigma_{3a}{}^b , \quad \sigma^2{}_a{}^b = -\sigma_{4a}{}^b , \quad \sigma^3{}_a{}^b = \sigma_{1a}{}^b$$

$$\sigma^4{}_a{}^b = -\sigma_{2a}{}^b , \quad \sigma^5{}_a{}^b = \sigma_{5a}{}^b$$

and according to (9.8) the only non-vanishing matrix elements of $\sigma_{ia}{}^b$ are

$$(\sigma_1)_2{}^1 = -(\sigma_1)_4{}^3 = \frac{1}{\sqrt{2}}\,, \quad (\sigma_2)_3{}^1 = (\sigma_2)_4{}^2 = -\frac{1}{\sqrt{2}}$$

$$(\sigma_3)_1{}^2 = -(\sigma_3)_3{}^4 = \frac{1}{\sqrt{2}}\,, \quad (\sigma_4)_1{}^3 = (\sigma_4)_2{}^4 = \frac{1}{\sqrt{2}}$$

$$(\sigma_5)_1{}^1 = -(\sigma_5)_2{}^2 = -(\sigma_5)_3{}^3 = (\sigma_5)_4{}^4 = -\frac{1}{2}\,.$$

Hence

$$(\psi^a \sigma_{ia}{}^b \psi_b)(\psi^c \sigma^i{}_c{}^d \psi_d) =$$
$$= (\psi^2 \sigma_{12}{}^1 \psi_1)(\psi^1 \sigma_{31}{}^2 \psi_2) + (\psi^2 \sigma_{12}{}^1 \psi_1)(\psi^3 \sigma_{33}{}^4 \psi_4) +$$
$$+ (\psi^4 \sigma_{14}{}^3 \psi_3)(\psi^1 \sigma_{31}{}^2 \psi_2) + (\psi^4 \sigma_{14}{}^3 \psi_3)(\psi^3 \sigma_{33}{}^4 \psi_4) + \ldots .$$

To be more precise we take ψ_a to be $|\{4\}, a>$ with weight $\underset{\sim}{m}_a$, so that the weight of ψ^a is $-\underset{\sim}{m}_a$. Then the weight of the product of two brackets, $(\psi^2 \sigma_{12}{}^1 \psi_1)(\psi^3 \sigma_{33}{}^4 \psi_4)$ say, is $-\underset{\sim}{m}_2 + \underset{\sim}{m}_1 - \underset{\sim}{m}_3 + \underset{\sim}{m}_4$ and Figure 30 shows that this is zero. In this way we see that any non-vanishing term (9.16) leads to a leptonic process (9.14) with

$$\underset{\sim}{m}_{f_1} + \underset{\sim}{m}_{f_4} = \underset{\sim}{m}_{f_2} + \underset{\sim}{m}_{f_3}\,;$$

that is, with charge and muon number conserved.

Combining (9.12), (9.13) and (9.17) we have that the Lagrangian density for a weak interaction process involving four particles, all of which are leptons or antileptons, is

$$c_1 (\bar{\psi}^a \gamma_\alpha (1+\gamma_5)\, \delta_a{}^b \psi_b)(\bar{\psi}^c \gamma_\alpha (1+\gamma_5)\, \delta_c{}^d \psi_d)$$
$$+ c_5 (\bar{\psi}^a \gamma_\alpha (1+\gamma_5)\, \sigma_{ia}{}^b \psi_b)(\bar{\psi}^c \gamma_\alpha (1+\gamma_5)\, \sigma^i{}_c{}^d \psi_d)$$
$$+ c_{10} (\bar{\psi} \gamma_\alpha (1+\gamma_5)\, T_{Aa}{}^b \psi_b)(\bar{\psi}^c \gamma_\alpha (1+\gamma_5)\, T^A{}_c{}^d \psi_d)\,.$$

The normalized ψ and $\bar{\psi}$ involve space, spin and group specifications, and c_1, c_5, c_{10} are independent constants. The possibility of occurrence of a process follows from the matrix elements of δ, σ and T.

As an example take an initial state with μ^- and e^+. In the product $\bar{\psi}^a \psi_b \bar{\psi}^c \psi_d$ we take ψ_b as annihilation of μ^- and $\bar{\psi}^a$ as annihilation of e^+, so that

$$a = 4 , \qquad b = 2 .$$

Since $\delta_4^{\,2}$ vanishes, there is no contribution from the one-dimensional representation. In the five-dimensional representation $\sigma_{i4}^{\;\;2}$ vanishes except for $i = 2$ when its value is $-\frac{1}{\sqrt{2}}$. The only non-vanishing elements of σ^2, i.e. $-\sigma_4$, are

$$(\sigma^2)_1^{\;3} = -\frac{1}{\sqrt{2}}, \qquad (\sigma^2)_2^{\;4} = -\frac{1}{\sqrt{2}} .$$

The product of the four fermion wave functions thus has one of the forms

$$(\bar{\psi}^4 \sigma_{24}^{\;\;2} \psi_2)(\bar{\psi}^1 \sigma^2{}_1^{\;3} \psi_3) = \frac{1}{2}$$

$$(\bar{\psi}^4 \sigma_{24}^{\;\;2} \psi_2)(\bar{\psi}^2 \sigma^2{}_2^{\;4} \psi_4) = \frac{1}{2} .$$

The $\bar{\psi}^1 \psi_3$ gives emission of ν_μ, $\bar{\nu}_e$ and $\bar{\psi}^2 \psi_4$ gives emission of μ^-, e^+ so the only processes allowed in this representation are

$$\mu^- + e^+ \rightarrow \begin{cases} \nu_\mu + \bar{\nu}\, e \\ \mu^- + e^+ \end{cases} \qquad (9.18)$$

and these occur with equal probabilities for weak interactions, if we assume the masses of the leptons to be equal. The assumption that the masses would be equal but for some symmetry-breaking interaction is implicit in the placing of the four leptons on a weight diagram. One should think of the scheme as having some validity in a high-energy region where rest masses are relatively unimportant. That the probabilities for the two processes in (9.18) are equal may be deduced also in the ten-dimensional representation from the non-vanishing

$$(\bar{\psi}^4\, T_{24}{}^2\, \psi_2)\,(\bar{\psi}^1\, T^2{}_1{}^3\, \psi_3) = -\frac{1}{2}$$

$$(\bar{\psi}^4\, T_{24}{}^2\, \psi_2)\,(\bar{\psi}^2\, T^2{}_2{}^4\, \psi_4) = \frac{1}{2}\,.$$

The one-dimensional representation comes into effect only for an initial state with zero weight, that is, for an initial particle-antiparticle state. For an initial electron-positron state we find that the one-, five-, and ten-dimensional representations give final states

$$e^- + e^+\,, \qquad \mu^- + \mu^+\,, \qquad \nu_e + \bar{\nu}_e \qquad \nu_\mu + \bar{\nu}_\mu$$

with equal probabilities.

CHAPTER X

Experimental Consequences of SU_3 Theory

The Supermultiplets

The most striking result of the application of the SU_3 group to strongly interacting particles is the existence of supermultiplets with the same spin and parity. They should also have equal masses and the actual differences of mass indicate the presence of interactions that do not preserve SU_3 invariance. We mentioned earlier that within an isotopic multiplet, e.g. π^+, π^0, π^-, the mass differences are ascribed to electromagnetic interactions. If these are neglected the interactions between the particles are strong, and the Lagrangian describing such interaction is invariant under the isospin SU_2 group. This invariance is just charge independence. To put it in a different way: strong interactions conserve the three components I_1, I_2, I_3 of isospin and consequently the total isospin. They also conserve the hypercharge Y. The electromagnetic interactions conserve I_3 and Y but not total isospin.

We have to distinguish two types of strong interactions. Very strong interactions are strong interactions that are invariant under the SU_3 group. They therefore give equal masses for all particles in a supermultiplet. Medium strong interactions are strong interactions that are not invariant under SU_3. To them are ascribed the mass differences between isospin multiplets in a supermultiplet, e.g. between π and K, between N and Σ. The medium strong interactions break the SU_3 symmetry in a manner analogous to that in which the electromagnetic interactions break the SU_2 symmetry. The medium strong interactions are supposed to be about ten times weaker than the very strong interactions and about ten times stronger than electromagnetic interactions.

The mass formula and other matters related to hadrons can be conveniently investigated by introducing the notion of U-spin. Having done this we shall consider separately the implications of group invariances for very strong, medium strong, electromagnetic and weak interactions.

U-Spin.

Let us rotate the m_1, m_2 axes of the weight diagrams through an angle $\frac{2\pi}{3}$ in the counter-clockwise direction obtaining the primed coordinates

$$m_1' = m_1 \cos\frac{2\pi}{3} + m_2 \sin\frac{2\pi}{3} = -\frac{1}{2}m_1 + \frac{\sqrt{3}}{2}m_2$$

$$m_2' = -m_1 \sin\frac{2\pi}{3} + m_2 \cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}m_1 - \frac{1}{2}m_2 \; .$$

Then

$$2\,m_2' = -\sqrt{3}\,m_1 - m_2 = -I_3 - \frac{1}{2}Y = -Q$$

$$\sqrt{3}\,m_1' = -\frac{\sqrt{3}}{2}m_1 + \frac{3}{2}m_2 = -\frac{1}{2}I_3 + \frac{3}{4}Y = U_z \, ,$$

where U_z is the eigenvalue of the third component of what we call U-spin. We shall also use U_z to denote the third component itself, the suffix z rather than 3 being employed to avoid confusion with the notation for the unitary group in three dimensions. We go from one position to another on the U_z-axis by operating with $E_{\pm 3}$. Now all particles on the U_z-axis have the same charge so, if

$$Q\,|a\rangle = \lambda\,|a\rangle \; ,$$

it will follow that

$$Q\,E_{\pm 3}\,|a\rangle = \lambda\,E_{\pm 3}\,|a\rangle = E_{\pm 3}\,\lambda\,|a\rangle = E_{\pm 3}\,Q\,|a\rangle$$

and therefore Q commutes with $E_{\pm 3}$. It commutes also with U_z because I_3 and Y commute with each other.

In the case of the octet one has to be careful about the two kets with zero weight. According to (8.6)

$$E_3 |\{8\},6\rangle = -\frac{1}{2\sqrt{3}} |\{8\},7\rangle + \frac{1}{2} |\{8\},8\rangle$$

$$E_3{}^2 |\{8\},6\rangle = -\frac{1}{3} |\{8\},3\rangle .$$

In terms of the ket vectors for the baryons, as we have related them at the end of Chapter VIII, the last equations may be written

$$E_3 \Xi^0 = \frac{1}{\sqrt{3}} \left(-\frac{1}{2}\Sigma^0 + \frac{\sqrt{3}}{2}\Lambda^0 \right)$$

$$E_3{}^2 \Xi^0 = -\frac{1}{3} n .$$

We therefore assign

$$\left(\Xi^0 , \ \tfrac{1}{2}\Sigma^0 - \frac{\sqrt{3}}{2}\Lambda^0 , \ n \right)$$

to the $U = 1$ triplet. The singlet $U = 0$ member of zero weight will then be $\frac{\sqrt{3}}{2}\Sigma^0 + \frac{1}{2}\Lambda^0$, which is orthogonal to $\frac{1}{2}\Sigma^0 - \frac{\sqrt{3}}{2}\Lambda^0$ and is annihilated by $E_{\pm 3}$. By making the usual replacement of symbols we can write down the singlet and triplet states of weight zero for the pseudoscalar and vector mesons. We put down (U_z , Q) diagrams for the octets and decuplet. Particles having the same charge belong to the same U-spin multiplet.

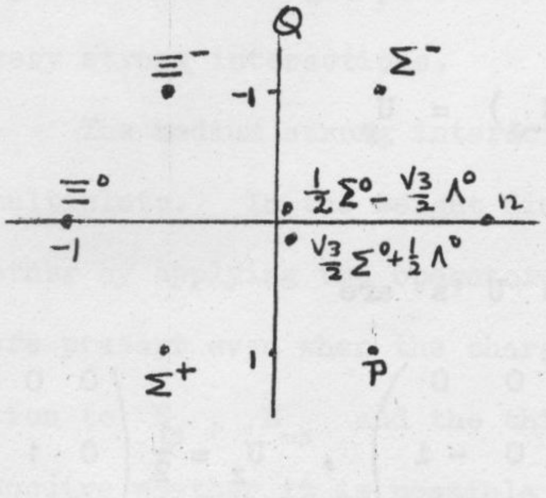

Fig. 33. The (U_z, Q) diagram for the baryon octet.

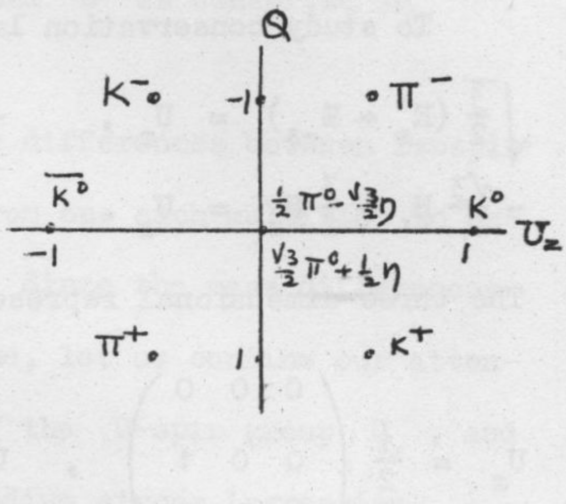

Fig. 34. The (U_z, Q) diagram for the pseudo-scalar meson octet.

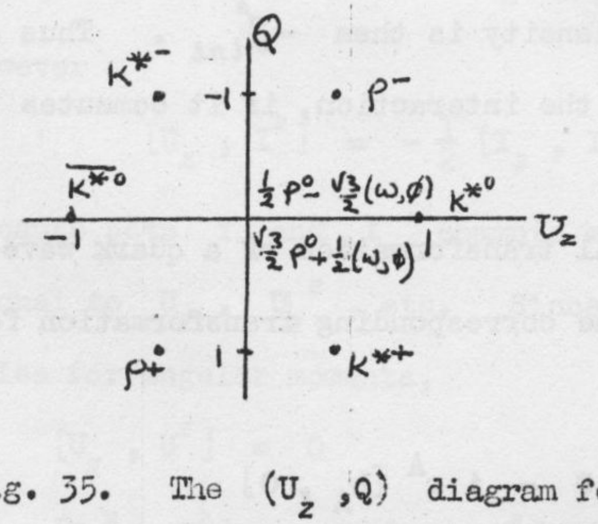

Fig. 35. The (U_z, Q) diagram for the vector meson octet. The symbol (ω^0, ϕ^0) denotes the combination of ω^0 and ϕ^0 that belongs to the octet.

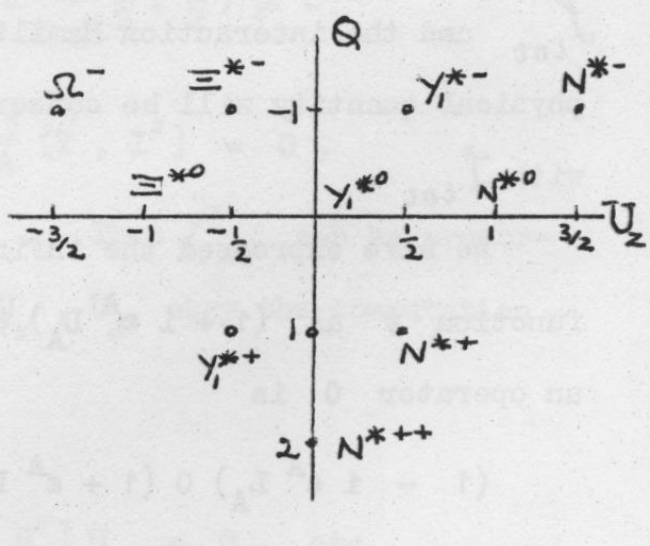

Fig. 36. The (U_z, Q) diagram for the baryon resonance $\frac{3}{2}$+ decuplet. The N* 's may also be denoted by Δ 's .

To study conservation laws we write

$$\sqrt{\frac{3}{2}}\,(E_3 + E_{-3}) = U_x , \qquad -\sqrt{\frac{3}{2}}\,i\,(E_3 - E_{-3}) = U_y$$

$$-\frac{\sqrt{3}}{2} H_1 + \frac{3}{2} H_2 = U_z .$$

The three-dimensional representation of the U 's are

$$U_x = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad U_y = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad U_z = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which shows that U_x , U_y , U_z as defined above are generators of the U-spin group. They commute with Q . In all strong and electromagnetic interactions U_z is conserved because I_3 and Y are separately conserved. We shall now look into the question as to whether total U is conserved. We assume a non-derivative interaction Lagrangian density $\mathcal{L}_{int}$ and the interaction Hamiltonian density is then $-\mathcal{L}_{int}$. Thus a physical quantity will be conserved in the interaction, if it commutes with $\mathcal{L}_{int}$.

We have expressed the infinitesimal transformation of a quark wave function ψ as $(1 + i\,\varepsilon^A L_A)\,\psi$. The corresponding transformation for an operator 0 is

$$(1 - i\,\varepsilon^A L_A)\, 0\, (1 + \varepsilon^A L_A) = 0 - i\,\varepsilon^A [L_A , 0] .$$

If 0 is invariant under SU_3 , it must commute with the eight independent operators of the group. The Lagrangian density for very strong interactions $\mathcal{L}_{vs}$ is by its very definition such an invariant operator. Hence $\mathcal{L}_{vs}$ commute with U_x , U_y , U_z , which are three independent operators of the group. It follows that total U is conserved in very strong interactions.

The medium strong interactions give mass differences between isospin multiplets. In the weight diagrams we go from one such multiplet to another by applying the operators $E_{\pm 2}$, $E_{\pm 3}$. Since the mass differences are present even when the charges are the same, let us confine our attention to E_3, E_{-3} and the third operator of the U-spin group U_z, and enquire whether it is possible to take the medium strong Lagrangian density $\mathcal{L}_{ms}$ proportional to these operators. Since total isospin is conserved, $\mathcal{L}_{ms}$ commutes with I^2. Now $E_{\pm 3}$ does not commute with I^2 because $E_{\pm 3}$ changes the I-eigenvalue; for example, in the baryon octet

$$[E_3 , I^2]\,\Sigma^+ = E_3\, 1.2\,\Sigma^+ - I^2 E_3 \Sigma^+$$
$$= -\frac{1}{\sqrt{6}}\left(1 . 2 - \frac{1}{2} . \frac{3}{2}\right) p .$$

However

$$[U_z , I^2] = -\frac{1}{2}[I_3 , I^2] + \frac{3}{4}[Y , I^2] = 0 ,$$

because both Y and I_3 commute with I^2. Thus $\mathcal{L}_{ms}$ can be proportional to U_z, $U_z^{\,2}$, etc. Since U_x, U_y, U_z obey the commutation rules for angular momenta,

$$[U_z , U^2] = 0$$

$$[U_z^{\,2} , U^2] = U_z [U_z , U^2] + [U_z , U^2]\, U_z = 0 , \text{ etc.}$$

Hence an expression for $\mathcal{L}_{ms}$ that is a linear combination of powers of U_z will conserve total U-spin. Nothing essentially new comes from considering E_2, E_{-2} and a third related operator.

To make the position of electromagnetic interactions clear we take the specific example of baryons interacting with the field of potential A_α, so that the Lagrangian density is

$$\mathcal{L}_{em} = i\,e\,(\overline{\Sigma^+}\gamma_\alpha \Sigma^+ + \bar{p}\gamma_\alpha p - \overline{\Sigma^-}\gamma_\alpha \Sigma^- - \overline{\Xi^-}\gamma_\alpha \Xi^-)\,A_\alpha ,$$

where γ_α denotes the Dirac matrices and units are chosen such that $\hbar = c = 1$. We write

$$b = \begin{pmatrix} \Sigma^+ \\ p \\ n \\ \Sigma^- \\ \Xi^- \\ \Xi^o \\ \Sigma^o \\ \Lambda^o \end{pmatrix}$$

for the baryon octet, where in SU_3 space $\Sigma^+ = |\{8\},1\rangle$, $p = |\{8\},2\rangle$ etc., and we see that

$$\mathcal{L}_{em} = i\,e\,(\bar{b}\,\gamma_\alpha\, Q\, b)\,A_\alpha .$$

Thus $\mathcal{L}_{em}$ transforms like Q and a similar result holds for quarks and for bosons, though there may be an additional Q^2 dependence. We have seen that Q commutes with U_x, U_y, U_z and therefore these three quantities and total U-spin are conserved in electromagnetic interactions. Since $\mathcal{L}_{em}$ is a scalar in U-space, it follows that the electromagnetic current which for example for baryons is $i\,e\,(\bar{b}\,\gamma_\alpha\, Q\, b)$, is a scalar. Moreover the photon, whose emission or absorption is given by the operator Q, should be treated as a scalar.

Very Strong Interactions

We wish to point out how conservation of U_z and of total U may provide information about branching ratios for the production of nuclear resonances by pion-nucleon and kaon-nucleon collisions(1). By inspection of Figures 34, 35, 36 we see that a $\pi^- - p$ collision can lead to $N^{*-}\rho^+$ and $Y_1^{*-}K^{*+}$, and that $\pi^- p$ belongs only to a $U = 1$ state since it is compounded of two $U = \frac{1}{2}$ states and has $U_z = 1$. The processes

$$\begin{aligned} \pi^- + p &\rightarrow N^{*-} + \rho^+ \\ \pi^- + p &\rightarrow Y_1^{*-} + K^{*+} \end{aligned} \tag{10.1}$$

can therefore come only from a $U = 1$ amplitude A, say. We recall the expansion of a state of total angular momentum number j and third component m

$$\Phi_{jm} = \sum_{j_1 j_2 m_1 m_2} (j_1\; j_2\; m_1\; m_2 \mid j\; m)\, \phi_{j_1 j_2 m_1 m_2},$$

where $(j_1\; j_2\; m_1\; m_2 \mid j\; m)$ are Clebsch-Gordan coefficients given by the formula

$$(j_1\; j_2\; m_1\; m_2 \mid j\; m) = \delta_{m, m_1+m_2} \left[\frac{(2j+1)\,(j_1+j_2-j)!\,(j+j_1-j_2)!\,(j+j_2-j_1)!}{(j_1+j_2+j+1)!} \right]^{\frac{1}{2}}$$

$$\times \left[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!\right]^{\frac{1}{2}}$$

$$\times \sum_{x=0} \frac{(-)^x}{x!} \left[(j_1+j_2-j-x)!\,(j_1-m_1-x)!\,(j_2+m_2-x)!\,(j-j_2+m_1+x)!\,(j-j_1-m_2+x)!\right]^{-1},$$

the summations being carried out as long as we have not the factorial of a

(1) S. Meskhov, C. A. Levinson and H. J. Lipkin, Phys. Rev. Lett. 10, 361 (1963).

negative integer. On reading off the values of U and U_z we see that the matrix elements for the processes (10.1) are

$$< \pi^- p \mid N^{*-} \rho^+ > = (\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2} - \tfrac{1}{2} \mid 1\,1) A = \frac{\sqrt{3}}{2} A$$

$$< \pi^- p \mid Y_1^{*-} K^{*+} > = (\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} \mid 1\,1) A = -\tfrac{1}{2} A .$$

Hence

$$|<\pi^- p \mid N^{*-} \rho^+ >|^2 = 3 \mid <\pi^- p \mid Y_1^{*-} K^{*+} >|^2 ,$$

which shows that the probability of occurrence of a final state $N^{*-} \rho^+$ is three times that of $Y_1^{*-} K^{*+}$.

Similarly, if we consider the processes

$$K^- + p \to Y_1^{*-} + \rho^+$$
$$K^- + p \to \Xi^{*-} + K^{*+} ,$$

we see from the diagrams that $K^- p$ has $U_z = 0$ and may belong to either $U = 0$ or $U = 1$, while the final states may both belong to $U = 1$ or $U = 2$. Thus only the $U = 1$ channel comes into play and

$$\frac{<K^- p \mid Y_1^{*-} \rho^+ >}{<K^- p \mid \Xi^{*-} K^{*+} >} = \frac{(\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2} - \tfrac{1}{2} \mid 1\,0)}{(\tfrac{3}{2}\tfrac{1}{2} - \tfrac{1}{2}\tfrac{1}{2} \mid 1\,0)} = 1 ;$$

so that there is equal probability for the two processes. These results presume that the SU_3 symmetry is exactly conserved; in other words, that everything apart from very strong interactions can be ignored.

Mass Relations

We saw that a possible medium strong Lagrangian density is a linear combination of powers of U_z. Let us examine the implications of taking $\mathcal{L}_{ms}$ proportional to U_z. In the Lagrangian density for free spin $\frac{1}{2}$ particles the mass multiplies a bilinear combination of wave functions;

for a proton field it is $-\bar{p}\,(\gamma_\alpha \frac{\partial}{\partial x_\alpha} + m_p)\,p$. The same seems to be generally true for fermions of higher spin[1]. On the other hand the mass squared appears for a boson field, e.g. $-m_\pi^{\,2}\,\phi^*\phi$ for the charged pseudoscalar field and $-\frac{1}{2}\,m_1^{\,2}\,\phi_\alpha\,\phi_\alpha$ for the neutral spin 1 field. Since the mass differences coming from $\mathcal{L}_{ms}$ are proportional to the eigenvalues of U_z , we write a mass operator

$$\mu = \alpha + \beta\, U_z \tag{10.2}$$

whose eigenvalues give the masses for a fermion supermultiplet and squares of the masses for a boson supermultiplet. The common mass of the members of the supermultiplet coming from the free Lagrangian density is accounted for by the α which like β is a scalar in U-spin space.

We apply (10.2) to the $U = 1$ multiplet of the baryon octet.

$$m(\Xi^0) = \langle \Xi^0 \,|\, m(\Xi^0) \,|\, \Xi^0 \rangle = \langle \Xi^0 \,|\, \mu \,|\, \Xi^0 \rangle$$

$$= \alpha + \beta \langle \Xi^0 \,|\, U_z \,|\, \Xi^0 \rangle = \alpha - \beta$$

$$m\,(\tfrac{1}{2}\,\Sigma^0 - \tfrac{\sqrt{3}}{2}\,\Lambda^0) = \alpha$$

$$m(n) = \alpha + \beta \; .$$

We are neglecting electromagnetic effects and therefore write

$$\alpha - \beta = m(\Xi) , \qquad \alpha + \beta = m(N)$$

$$\alpha = \frac{1}{4}\,m(\Sigma) + \frac{3}{4}\,m(\Lambda) ,$$

the $\frac{1}{2}\,\Sigma^0 - \frac{\sqrt{3}}{2}\,\Lambda^0$ state having probability $\frac{1}{4}$ of being Σ^0 and $\frac{3}{4}$ of being Λ^0 . Hence

$$\frac{3\,m(\Lambda) + m(\Sigma)}{4} = \frac{m(\Xi) + m(N)}{2} ,$$

(1) W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

a relation satisfied by the experimental values of the masses to within 2 per cent. It may easily be checked that this relation satisfies the Okubo formula (6.2)

$$M = a + b\,Y + c\,\{\tfrac{1}{4}Y^2 - I\,(I+1)\} .$$

For the pseudoscalar mesons we have similarly

$$\frac{3\,(m\,(\eta))^2 + (m\,(\pi))^2}{4} = \frac{(m\,(K))^2 + (m\,(\bar{K}))^2}{2} = (m\,(K))^2 ,$$

which is verified reasonably well by experiment.

For the $\frac{3}{2}+$ decuplet we re-write (10.2) as

$$\mu = \alpha' + \beta'\,U_z ,$$

because the constants may depend on the representation. The $U = \frac{3}{2}$ multiplet in Figure 36 shows that

$$m\,(\Omega^-) = \alpha' - \tfrac{3}{2}\beta' , \qquad m\,(\Xi^*) = \alpha' - \tfrac{1}{2}\beta'$$

$$m\,(Y_1^*) = \alpha' + \tfrac{1}{2}\beta' , \qquad m\,(N^*) = \alpha' + \tfrac{3}{2}\beta' ,$$

and therefore

$$m\,(\Omega^-) - m\,(\Xi^*) = m\,(\Xi^*) - m\,(Y_1^*) = m\,(Y_1^*) - m\,(N^*) . \quad (10.3)$$

Figure 18 shows that

$$Y = 2\,(I - 1)$$

and the Okubo formula reduces to

$$M = a' + b'\,Y ,$$

which gives the equalities (10.3) for the mass differences that are found experimentally. If we add a term proportional to $U_z^{\,2}$ to the expression for the mass operator μ , we can get a correction to the original mass formula (6.2)[1].

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(1) H. Harari, Nuovo Cimento 33, 752 (1964).

So far we have said nothing about mass differences within isospin multiplets. Since they all have the same I and the same Y, the Okubo formula will not distinguish between their masses. Take four points p, q, r, s on a weight diagram as shown in Figure 37. If electromagnetic interactions are neglected, the masses of the particles in the same isospin multiplet are equal and so

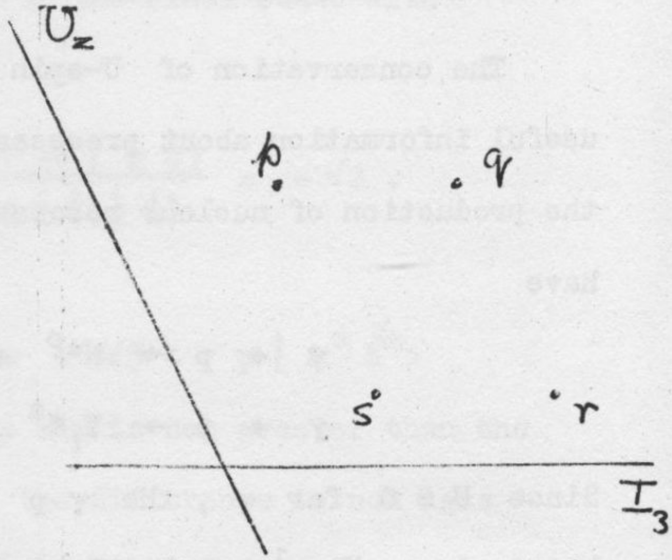

Fig. 37. Weights at the corners of a parallelogram in (I_3, U_z) space.

$$m(p) = m(q) , \quad m(s) = m(r) . \tag{10.4}$$

On the other hand, if medium strong interactions are neglected, the particles s and p having the same unperturbed mass and the same charge will have the same mass when the electromagnetic interaction is switched on, so that

$$m(s) = m(p) , \quad m(r) = m(q) . \tag{10.5}$$

Equations (10.4) and (10.5) may be comprised in

$$m(p) - m(q) + m(r) - m(s) = 0 ,$$

a relation which being obeyed separately by (10.4) and (10.5) holds for all orders in $\mathcal{L}_{ms}$ and $\mathcal{L}_{em}$, provided that we neglect interference between the two kinds of interaction. From the decuplet we obtain

$$m(N^{*-}) - m(N^{*0}) + m(Y_1^{*0}) - m(Y_1^{*-}) = 0$$

$$m(N^{*0}) - m(N^{*+}) + m(Y_1^{*+}) - m(Y_1^{*0}) = 0$$

$$m(Y_1^{*-}) - m(Y_1^{*0}) + m(\Xi^{*0}) - m(\Xi^{*-}) = 0 ,$$

for which there is some experimental evidence.

<u>Electromagnetic Interactions</u>

The conservation of U-spin for electromagnetic interactions provides useful information about processes in which photons take part[1]. Consider the production of nuclear resonances by photon-proton collisions. We can have

$$\gamma + p \rightarrow N^{*0} + \pi^{+}$$
$$\gamma + p \rightarrow Y_1^{*0} + K^{+}.$$

Since $U = 0$ for γ, the $\gamma\, p$ state has $U = \frac{1}{2}$, $U_z = \frac{1}{2}$, so that we have only a $U = \frac{1}{2}$ amplitude in both cases. On inspection of Figures 33-36 we see that

$$\frac{\langle \gamma\, p \mid N^{*0} \pi^{+} \rangle}{\langle \gamma\, p \mid Y_1^{*0} K^{+} \rangle} = \frac{(1\ \frac{1}{2}\ 1\ {-\frac{1}{2}} \mid \frac{1}{2}\ \frac{1}{2})}{(1\ \frac{1}{2}\ 0\ \frac{1}{2} \mid \frac{1}{2}\ \frac{1}{2})} = -\sqrt{2}$$

and similarly

$$\frac{\langle \gamma\, p \mid N^{*0} \rho^{+} \rangle}{\langle \gamma\, p \mid Y_1^{*0} K^{*+} \rangle} = -\sqrt{2}$$

$$\frac{\langle \gamma\, n \mid N^{*-} \pi^{+} \rangle}{\langle \gamma\, n \mid Y_1^{*-} K^{+} \rangle} = \frac{\langle \gamma\, n \mid N^{*-} \rho^{+} \rangle}{\langle \gamma\, n \mid Y_1^{*-} K^{*+} \rangle} = \frac{(\frac{3}{2}\ \frac{1}{2}\ \frac{3}{2}\ {-\frac{1}{2}} \mid 1\ 1)}{(\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\quad 1\ 1)} = -\sqrt{3}\,.$$

When the baryons Λ^0 and Σ^0 are produced, our information may be less precise. Suppose that we wish to compare the amplitudes for

$$\gamma + p \rightarrow n + \pi^{+}$$
$$\gamma + p \rightarrow \Lambda^0 + K^{+}$$
$$\gamma + p \rightarrow \Sigma^0 + K^{+}.$$

The final state with n and π^{+} is obtained by compounding $U = 1$ and

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(1) C. A. Levinson, H. J. Lipkin and S. Meshkov, Phys. Lett. 7, 81 (1963).

$U = \frac{1}{2}$, and the same composition arises from the final state with $\frac{1}{2}\Sigma^0 - \frac{\sqrt{3}}{2}\Lambda^0$ and K^+, so

$$\frac{\langle \gamma\, p \mid n\,\pi^+ \rangle}{\langle \gamma\, p \mid \frac{1}{2}\Sigma^0 - \frac{\sqrt{3}}{2}\Lambda^0, K^+ \rangle} = \frac{(1\,\frac{1}{2}\,1\,-\frac{1}{2} \mid \frac{1}{2}\,\frac{1}{2})}{(1\,\frac{1}{2}\,0\,\frac{1}{2} \mid \frac{1}{2}\,\frac{1}{2})} = -\sqrt{2}\,.$$

Hence

$$\sqrt{2}\,\langle \gamma\, p \mid n\,\pi^+ \rangle = -\langle \gamma p \mid \Sigma^0 K^+ \rangle + \sqrt{3}\,\langle \gamma\, p \mid \Lambda^0 K^+ \rangle$$

and using theorems that the modulus of the sum is not greater than the sum of the moduli and that the modulus of the difference is not less than the modulus of the difference of the moduli we deduce the inequalities

$$|\langle \gamma\, p \mid \Sigma^0 K^+ \rangle + \sqrt{3}\,|\langle \gamma\, p \mid \Lambda^0 K^+ \rangle| \geq \sqrt{2}\,|\langle \gamma\, p \mid n\,\pi^+ \rangle|$$

$$\geq \big|\,|\langle \gamma\, p \mid \Sigma^0 K^+ \rangle| - \sqrt{3}\,|\langle \gamma\, p \mid \Lambda^0 K^+ \rangle|\,\big|\,.$$

Inspection of the $(U_z\,, Q)$ diagrams also gives information about electromagnetic decays; for example, while

$$Y_1^{*+} \to \Sigma^+ + \gamma$$

is allowed,

$$Y_1^{*-} \to \Sigma^- + \gamma$$

is forbidden. In fact Y_1^{*-} belongs to a $U = 3/2$ state but $\Sigma^-\gamma$ belongs to $U = \frac{1}{2}$ and conservation of total U forbids the decay. It is also obvious that the decay

$$N^{*+} \to p + \gamma$$

is allowed.

We can compare magnetic moments under the drastic assumption that medium strong interactions can be neglected while we take account of electromagnetic interactions. Then all members of a U-spin multiplet have the same mass and charge, and therefore the same magnetic moment μ.

This gives

$$\mu(p) = \mu(\Sigma^+) , \quad \mu(\Xi^o) = \mu(n) , \quad \mu(\Xi^-) = \mu(\Sigma^-) .$$

Weak Interactions.

To facilitate comparison with the literature on weak interactions we write down the three-dimensional Hermitian matrix representation of the SU_3 operators as given by Gell-Mann[1]

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

They have the properties

$$\mathrm{tr}\, \lambda_i \lambda_j = 2\, \delta_{ij} , \quad [\lambda_i , \lambda_j] = 2\, i\, f_{ijk}\, \lambda_k ,$$

$$\{\lambda_i , \lambda_j\} = 2\, d_{ijk}\, \lambda_k + \frac{4}{3} \delta_{ij}\, 1 ,$$

where $\{ \}$ denotes anticommutator, 1 is the unit matrix, f_{ijk} is real and totally antisymmetric, d_{ijk} is real and totally symmetric in i , j , k. The values of f_{ijk} and d_{ijk} are tabulated in Gell-Mann's paper. We see that

$$\begin{aligned} \lambda_1 + i\,\lambda_2 &= 2\sqrt{6}\, E_1 , \quad \lambda_4 + i\,\lambda_5 = 2\sqrt{6}\, E_2 , \\ \lambda_6 + i\,\lambda_7 &= 2\sqrt{6}\, E_3 , \quad \lambda_3 = 2\sqrt{3}\, H_1 , \quad \lambda_8 = 2\sqrt{3}\, H_2 . \end{aligned} \tag{10.6}$$

(1) M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

According to (6.4) the I_3-operator is $\sqrt{3}\,H_1$ and the Y-operator is $2\,H_2$, so the charge operator is

$$Q = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 . \tag{10.7}$$

The usual approach to the study of weak interactions is to assume that the Lagrangian density for such interactions is of the current-current type $J_\alpha^{+}\mathcal{J}_\alpha$ summed over α from 1 to 4. Suppose that the free field is described by the variational principle

$$\delta\int \mathcal{L}\left(\psi_\mu , \frac{\partial\psi_\mu}{\partial x_\alpha}\right) d^3x\,dt = 0 . \tag{10.8}$$

Then the Eulerian equations are

$$\frac{\partial\mathcal{L}}{\partial\psi_\mu} - \sum_\mu \frac{\partial}{\partial x_\alpha}\frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x_\alpha}} = 0 . \tag{10.9}$$

Let the variation in ψ_μ be denoted by $\delta\psi_\mu$, so that

$$\delta\mathcal{L} = \sum_{\mu,\alpha}\left\{\frac{\partial\mathcal{L}}{\partial\psi_\mu}\delta\psi_\mu + \frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x_\alpha}}\delta\left(\frac{\partial\psi_\mu}{\partial x_\alpha}\right)\right\}, \tag{10.10}$$

and consider f_α defined by

$$f_\alpha = \sum_\mu \frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x_\alpha}}\delta\psi_\mu . \tag{10.11}$$

We see that

$$\sum_\alpha \frac{\partial f_\alpha}{\partial x_\alpha} = \sum_{\mu,\alpha}\left\{\frac{\partial}{\partial x_\alpha}\frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x_\alpha}}\delta\psi_\mu + \frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x_\alpha}}\frac{\partial}{\partial x_\alpha}(\delta\psi_\mu)\right\}$$

$$= \sum_{\mu,\alpha}\left\{\frac{\partial\mathcal{L}}{\partial\psi_\mu}\delta\psi_\mu + \frac{\partial\mathcal{L}}{\partial\dfrac{\partial\psi_\mu}{\partial x}}\delta\left(\frac{\partial\psi_\mu}{\partial x_\alpha}\right)\right\}, \text{ by (10.9)}$$

$= \delta\mathcal{L}$, by (10.10).

For (10.8) to be true for an arbitrary integration region $\delta\mathcal{L}$ must be zero and it follows that f_α has vanishing four-divergence. The free Lagrangian density for fermions of equal mass taking part in decay processes is

$$\mathcal{L} = -\bar{\psi}\left(\gamma_\alpha \frac{\partial}{\partial x_\alpha} + \kappa\right)\psi ,$$

where κ is the bare mass and $\hbar = c = 1$. Let the infinitesimal transformation of ψ be

$$\psi_l \rightarrow (\delta_l^{\ m} + i\,\varepsilon^A X_{Al}^{\ \ m})\,\psi_m .$$

Then (10.11) shows that the vector current density $(\bar{\psi}\gamma_\alpha X_A \psi)$ is conserved, if we neglect mass differences.

To discuss leptonic decays of strongly interacting particles we define F-spin current density $\mathcal{F}_{i\alpha}(x)$ by

$$\mathcal{F}_{i\alpha} = \frac{i}{2}\,\bar{q}\,\lambda_i\,\gamma_\alpha\,q ,$$

where q is the quark wave function. $\mathcal{F}_{i\alpha}$ is a conserved vector current density with respect to α and an SU_3 eight-vector with respect to i . Then

$$\begin{aligned} \mathcal{F}_{3\alpha} + \frac{1}{\sqrt{3}}\mathcal{F}_{8\alpha} &= \frac{i}{2}\,\bar{q}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right)\gamma_\alpha\,q \\ &= i\,\bar{q}\,\gamma_\alpha\,Q\,q, \text{ by } (10.7) , \end{aligned}$$

and so is the electromagnetic current density in units of the proton charge. On account of this result we are led to make the assumption that the weak vector current densities responsible for decays are also combinations of the eight $\mathcal{F}_{i\alpha}$'s .

Next consider $\mathcal{F}_{1\alpha} + i\,\mathcal{F}_{2\alpha}$, that is $\sqrt{6}\,i\,\bar{q}\,\gamma_\alpha\,E_1\,q$ according to (10.6). We recall that the operator E_1 increases m_1 by $\frac{1}{\sqrt{3}}$ and leaves m_2 unchanged; that is to say, it produces a transition with

$$\Delta I_3 = 1 , \qquad \Delta Y = 0$$

and therefore

$$\Delta Q = 1 , \qquad \Delta S = 0 ,$$

because Y is the sum of S and baryon number, which is conserved. Then $\mathcal{J}_{4\alpha} + i\mathcal{J}_{5\alpha}$ produces a transition with

$$\Delta I_3 = \tfrac{1}{2} , \qquad \Delta Y = 1$$

and therefore

$$\Delta Q = 1 , \qquad \Delta S = 1 .$$

The current density $\mathcal{J}_{6\alpha} + i\mathcal{J}_{7\alpha}$ would give rise to

$$\Delta I_3 = -\tfrac{1}{2} , \qquad \Delta Y = 1 , \qquad \Delta Q = 0 , \qquad \Delta S = 1 .$$

Such decays do not occur in nature. Thus we have two conserved vector current densities

$$\mathcal{J}_{1\alpha} + i\mathcal{J}_{2\alpha} , \qquad \mathcal{J}_{4\alpha} + i\mathcal{J}_{5\alpha} , \tag{10.12}$$

the first for decays with $\Delta I_3 = \Delta Q = 1$, $\Delta S = 0$ and the second with $\Delta I_3 = \frac{1}{2}$, $\Delta Q = \Delta S = 1$.

We can now establish a Lagrangian density $\mathcal{L}_W$ for the decay of hadrons. With the current-current hypothesis

$$\mathcal{L}_W = \frac{G}{\sqrt{2}} J_\alpha^{+} \mathcal{J}_\alpha ,$$

where

$$J_\alpha = J_{L\alpha} + \mathcal{J}_{H\alpha} ,$$

L denoting the leptons and H the hadrons. We already noted that

$$J_{L\alpha} = i\,(\bar{\nu}_e \gamma_\alpha (1 + \gamma_5)\, e) + i\,(\bar{\nu}_\mu \gamma_\alpha (1 + \gamma_5)\, \mu) .$$

The factor $(1 + \gamma_5)$ is inserted in order to have the leptons left-hand polarized. $J_{L\alpha}$ is the sum of a conserved vector current density

$i(\bar{\nu}_e \gamma_\alpha e) + i(\bar{\nu}_\mu \gamma_\alpha \mu)$ and an axial vector current density $i(\bar{\nu}_e \gamma_\alpha \gamma_5 e) + i(\bar{\nu}_\mu \gamma_\alpha \gamma_5 \mu)$ that is not conserved. In analogy with this we add to (10.12) axial vector current densities

$$\mathcal{F}_{1\alpha}^{(5)} + i\,\mathcal{F}_{2\alpha}^{(5)}\,, \quad \mathcal{F}_{4\alpha}^{(5)} + i\,\mathcal{F}_{5\alpha}^{(5)}$$

and assume that $\mathcal{F}_{i\alpha}^{(5)}$ transforms like $\mathcal{F}_{i\alpha}$ under the group. Then we write

$$J_{H\alpha} = [\mathcal{F}_{1\alpha} + \mathcal{F}_{1\alpha}^{(5)} + i(\mathcal{F}_{2\alpha} + \mathcal{F}_{2\alpha}^{(5)})]\cos\theta + {} + [\mathcal{F}_{4\alpha} + \mathcal{F}_{4\alpha}^{(5)} + i(\mathcal{F}_{5\alpha} + \mathcal{F}_{5\alpha}^{(5)})]\sin\theta\,, \qquad (10.13)$$

where the value of the Cabibbo angle θ is somewhere between .2 and .26 radians[1]. $\mathcal{L}_W$ then gives for leptonic decays the rules $\Delta I_3 = \Delta Q = \pm 1$, $\Delta S = 0$ and $\Delta I_3 = \pm\frac{1}{2}$, $\Delta Q = \Delta S = \pm 1$, the ambiguity in sign arising from the fact that $\mathcal{L}_W$ is the product of $\mathcal{J}_\alpha^+$ and $\mathcal{J}_\alpha$. In none of the above cases do we get $\Delta S = -\Delta Q$, $\Delta I_3 = -3/2$, such as occurs in $\Sigma^+ \rightarrow n + e^+ + \nu_e$.

If we make the transformations

$$\lambda_1 \cos\theta + \lambda_4 \sin\theta = \lambda_1', \quad \lambda_2 \cos\theta + \lambda_5 \sin\theta = \lambda_2'$$

with consequent transformations of $\mathcal{F}_{i\alpha}$ and $\mathcal{F}_{i\alpha}^{(5)}$, we can express (10.13) as

$$J_{H\alpha} = \mathcal{F}_{1\alpha}' + \mathcal{F}_{1\alpha}^{(5)\prime} + i(\mathcal{F}_{2\alpha}' + \mathcal{F}_{2\alpha}^{(5)\prime}).$$

The non-leptonic part of $\mathcal{L}_W$ is

$$\mathcal{L}_{NL} = \frac{G}{\sqrt{2}}\{(\mathcal{F}_{1\alpha}' + \mathcal{F}_{1\alpha}^{(5)\prime})(\mathcal{F}_{1\alpha}' + \mathcal{F}_{1\alpha}^{(5)\prime}) + (\mathcal{F}_{2\alpha}' + \mathcal{F}_{2\alpha}^{(5)\prime})(\mathcal{F}_{2\alpha}' + \mathcal{F}_{2\alpha}^{(5)\prime})\}$$

This is the sum of products of two eight vectors and, since

$$D^{(8)}(1,1) \otimes D^{(8)}(1,1) = D^{(1)}(0,0) \oplus D^{(8)}(1,1) \oplus D^{(8)}(1,1) \oplus D^{(10)}(3,0) \oplus D^{(10)}(0,3) \oplus D^{(27)}(2,2)\,,$$

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(1) N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963); J. J. Sakurai, ibid. 12, 79 (1964).

$\mathcal{L}_{NL}$ can be the sum of vectors in several different dimensions. The assumption is generally made that in the product representation the octet gives the predominant contribution. Then $\mathcal{L}_{NL}$ is taken to transform like the member of an octet[1].

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(1) N. Cabibbo, Phys. Rev. Lett. 12, 62 (1964).